1.1. For which real numbers a does the equation

$$\frac{\partial u}{\partial t}(x,t) = (x^2+t^2)^a$$

have a solution in each of the following three domains:

$$\begin{aligned}\Omega_1 &= \{(x,t)\in\mathbf{R}^2;\ t>0 \text{ or } x\neq 0\},\\ \Omega_2 &= \{(x,t)\in\mathbf{R}^2;\ x>0 \text{ or } t\neq 0\},\\ \Omega_3 &= \{(x,t)\in\mathbf{R}^2;\ (x,t)\neq(0,0)\}.\end{aligned}$$

1.2. Study the solvability of the equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{x}{x^2+t^2}$$

in the domain

$$\Omega_2 = \{(x,t)\in\mathbf{R}^2;\ x>0 \text{ or } t\neq 0\},$$

as well as in

$$\Omega_3 = \{(x,t)\in\mathbf{R}^2;\ (x,t)\neq(0,0)\}.$$

1.3. Find an integral curve to the system

$$\frac{dx}{yz} = \frac{dy}{z} = \frac{dz}{2x-y^2}$$

which passes through the point $(1,1,1)$.

1.4. Find an integral curve to the system

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

which passes through the point $(1,1,-2)$.

1.5. Find an integral curve to the equation

$$z_x + 2z_y = 0$$

which passes through the curve

$$t \mapsto (t+t^2, 2t^2, t^2)$$

in (x,y,z)-space, i.e., find a function u of (x,y) such that

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

and such that $u(t+t^2, 2t^2) = t^2$ for t real.

2.1. Find all solutions to the following equations:

(a) $$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0;$$

(b) $$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0;$$

(c) $$y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0;$$

(d) $$y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u;$$

(e) $$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0;$$

(f) $$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = x;$$

(g) $$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = x + y.$$

2.2. Solve the Cauchy problem

$$\frac{\partial u}{\partial x} + (x+y)\frac{\partial u}{\partial y} = 1,$$
$$u(x, -x) = 0.$$

2.3. Solve the Cauchy problem

$$(1+x^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 0,$$
$$u(x, x+x^3) = h(x).$$

2.4. Find a function u defined in some open neighborhood of the x-axis in $\mathbf{R}^2$ such that

$$x^2\frac{\partial u}{\partial x} + (y+1)\frac{\partial u}{\partial y} = 0,$$
$$u(x, 0) = x.$$

Prove that if u is a solution to the differential equation in the whole plane, then $u(x,0)$ is constant for $x \geqslant 0$. By way of contrast, $u(x,0)$ need not to be constant for $x < 0$, but the limit $\lim_{x\to-\infty} u(x,0)$ exists.

2.5 A velocity field $(u, v)\colon \mathbf{R}^2 \to \mathbf{R}^2$ is given by $u(x,y) = 2xy$, $v(x,y) = 1 + x^2 - y^2$. Determine the streamlines.

3.1. Prove that the only solutions in all of $\mathbf{R}^2$ to the equation

$$u^3\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

are the constants.

3.2. Solve the Cauchy[1] problem

$$(y+1)\frac{\partial u}{\partial x}+(x+1)\frac{\partial u}{\partial y}=u^2,$$

with a solution surface containing the curve $(s,-s,1/\log s)$, $s>0$.

3.3. Find a solution u to the equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

which is defined for $x\geqslant 2$ and which satisfies $u(2,y)=y^2+1$.

3.4. Find a solution u to the equation

$$x^2\frac{\partial u}{\partial x}-y^2\frac{\partial u}{\partial y}+2(x-y)u=0$$

which satisfies $u(x,x)=x$.

3.5. Prove that the initial-value problem

$$\begin{cases} x\dfrac{\partial u}{\partial x}+t\dfrac{\partial u}{\partial t} &= u^3,\\ u(x,0) &= x,\end{cases}$$

has no solution.

3.6. Solve the Cauchy problem

$$u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=1+u^2,$$

so that the solution surface contains the curve $(s,-s,\tan s)$ for small values of $|s|$.

3.7. Solve the initial-value problem

$$\begin{cases} u^2\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial t} &= u,\\ u(x,0) &= x.\end{cases}$$

[1] Augustin Louis Cauchy (1789–1857).

4.1. Solve the initial-value problem

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial y}\right)^2, \qquad u(0,y) = y^2/2,$$

using the method of envelopes as well as the method of characteristic strips. Compare the two methods of computing.

4.2. Solve the Cauchy problem

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 1, \qquad u(x,-x) = 1.$$

How many solutions are there?

4.3. Solve the initial-value problem

$$\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} = u, \qquad u(s,0) = s.$$

4.4. Solve the initial-value problem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = u, \qquad u(s,0) = s^2.$$

Try both characteristic strips and envelopes of affine solutions. Which method is the easiest in this case?

4.5. Solve the Cauchy problem

$$\left(\frac{\partial u}{\partial x}\right)^2 + 4y\left(\frac{\partial u}{\partial y}\right)^2 = 2, \qquad u(s,1) = s+1.$$

How many solutions are there?

4.6. Find the solution to the equation

$$x\left(\frac{\partial u}{\partial x}\right)^2 + y\frac{\partial u}{\partial y} = 0$$

which satisfies $u(s,1) = -s$.

4.7. Solve the Cauchy problem

$$x\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^3 = 1, \qquad u(s,0) = \sqrt{s}, \quad s > 0.$$

5.1. Determine all characteristic curves to the equation

$$\frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Transform the equation to normal form in the open set $y \neq 0$.

5.2. Determine the characteristic curves to the equation

$$\frac{\partial^2 u}{\partial x^2} - 9x^4 \frac{\partial^2 u}{\partial y^2} - 6xu = 0.$$

Transform the equation to normal form in the domain $x > 0$.

5.3. Determine the characteristic curves to the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - 2x \frac{\partial^2 u}{\partial x \partial y} + \frac{3}{4} \frac{\partial^2 u}{\partial y^2} + \frac{1}{2} \frac{\partial u}{\partial y} = 0.$$

Transform the equation to normal form in all of $\mathbf{R}^2$. Find the general solution.

5.4. Solve the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

with the initial-value conditions $u(x,0) = x$, $u_t(x,0) = 0$. *Hint:* Experiment with exponential functions multiplied by solutions to the ordinary wave equation.

5.5. Solve the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

in the domain $t > 0$, $x > 0$ with the boundary conditions

$$u(x,0) = x^2, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(0,t) = 0.$$

Hint: Think about even and odd functions.

5.6. Solve the initial-value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right)$$

with the conditions

$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 1, & |x| \leqslant a \\ 0, & |x| > a. \end{cases}$$

Discuss the behavior of the solution at the point $(0, a/c)$; it is not continuous there, a fact which depends on the discontinuities in the initial conditions on the sphere $t = 0$, $|x| = a$. The phenomenon is called the focusing effect. *Hint:* Show first that u is a function of $(|x|, t) = (r, t)$ and that $v = ru$ solves the wave equation in the variables (r, t). (This is special for three space variables.)

6.1. Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{ when } t > ax; \qquad u(x,t) = 0 \text{ and } \frac{\partial u}{\partial t}(x,t) = x \text{ when } t = ax,$$

where a is a constant $\neq \pm 1/c$. *Hint:* Introduce new coordinates with the help of the Lorentz[2] transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}},$$

for a suitable v, or, even simpler,

$$x' = x - vt, \quad t' = t - vx/c^2.$$

6.2. Solve the Dirichlet[3] problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in the square } 0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi,$$

with the boundary values $u(x,0) = u(0,y) = u(\pi,y) = 0$ and $u(x,\pi) = \sin mx$.

6.3. Let g be a function which is continuous on $\mathbf{R}$ and satisfies $g(x) \leqslant C/(1+x^2)$. Define a function u on $\mathbf{R}^2$ as

$$u(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \log((x_1 - y_1)^2 + x_2^2) g(y_1) dy_1.$$

Prove that $\Delta u = 0$ when $x_2 > 0$ and that $u_{x_2}(x_1, x_2) \to g(x_1)$ as $x_2 \to 0$.

6.4. Decide whether the following statements are true or false.
a) If u is a harmonic function in the plane, then e^u is subharmonic.
b) If u is a harmonic function in the plane, then $\log(1+u^2)$ is subharmonic.

6.5. The potential of a mass distribution with constant surface density on a sphere is defined by

$$u(x) = \gamma \int_{\|y\|=r} \frac{1}{\|x-y\|} dS(y), \qquad x \in \mathbf{R}^3,$$

where γ is a positive constant. Determine u.

6.6. Prove that if u is a bounded solution to $\Delta u = u$ in all of $\mathbf{R}^n$, then u must be zero. *Hint:* Use the translation invariance and the rotation invariance of the operator and then compare it with the solution of, e.g., $v(x) = a + b\|x\|^2$, where the constants a and b are chosen so that $0 \leqslant \Delta v \leqslant v$.

[2] Hendrik Antoon Lorentz (1853–1928) $\neq$ Edward Norton Lorenz (b. 1917-05-23). The transformations named after the former are used in relativity theory; the latter is known for his attractor and the butterfly effect.

[3] Peter Gustav Lejeune Dirichlet (1805–1859).

From McOwen, page 90:

7.1 Find the solution to the intitial-value problem $u_{tt} = u_{xx}+u_{yy}+u_{zz}$, $u(x, y, z, 0) = x^2 + y^2$, $u_t(x, y, z, 0) = 0$ using Kirchhoff's formula. Then notice that the initial values do not depend on z, the third space coordinate, and solve the problem using the formula obtained in two space variables by Hadamard's method of descent from Kirchhoff's formula in three variables.

7.2 Use Duhamel's principle to find the solution to the nonhomogeneous wave equation in three space dimensions $u_{tt} - c^2\Delta u = f(x, t)$ with initial conditions $u(x, 0) = 0$, $u_t(x, 0) = 0$. What regularity in f is required for the solution to be in C^2?

From McOwen, page 99:

7.3. Find dispersive wave solutions of the n-dimensional linear Klein–Gordon equation $u_{tt} - c^2\Delta u + m^2u = 0$.

7.4. Show that each of the following linear equations has dispersive wave solutions $u(x, t) = \exp\big(i(kx - \omega t)\big)$, $(x, t) \in \mathbf{R}^2$:
(a) The flexible beam equation $u_{tt} + \gamma^2 u_{xxxx} = 0$;
(b) The linearized Korteweg–de Vries equation $u_t + cu_x + u_{xxx} = 0$;
(c) The Boussinesq equation $u_{tt} - c^2u_{xx} = \gamma^2 u_{ttxx}$;
(d) The Schrödinger equation $u_t = i\Delta u$.

7.5. Show that the heat equation $u_t = u_{xx}$ admits uniform wave solutions of the form $U(kx - \omega t) = e^{i(kx-\omega t)}$ in which ω is a complex number and the wave is exponentially decaying in t. (Such uniform waves are called *diffusive*.) Prove that there are no waves without attenuation which are bounded when $t = 0$.

7.6. Find two uniform wave solutions of the equation $u_{tt} - u_{xx} + \lambda u = 0$ with $\lambda > 0$ satisfying the initial condition $u(x, 0) = 3\cos 2x$.

7.7. Find a condition on u_0 and u_1 that is necessary for the existence of a uniform wave solution of $u_{tt} - u_{xx} + \lambda u = 0$ satisfying the initial conditions $u(x, 0) = u_0(x)$ and $u_t(x, 0) = u_1(x)$.

7.8. Find the solution of the telegrapher's equation $u_{tt} - u_{xx} + u_t + m^2u = 0$ satisfying the initial conditions $u(x, 0) = u_0(x)$ and $u_t(x, 0) = 0$, where u_0 is an arbitrary C^2 function on the real axis.

8.1. Rewrite the initial-value problem

$$u_{tt} = c^2 u_{xx}; \quad u(x,0) = f(x),\ u_t(x,0) = g(x) \text{ when } x \in \mathbf{R},$$

as an initial-value problem for the vector-valued function $(v_1, v_2)^{\mathrm{T}} = (u_t, u_x)^{\mathrm{T}}$. Reduce it to the canonical form $v_t + Bv_x = Cv + D$ with a diagonal matrix B.

8.2. Use the canonical form obtained in 8.1 to solve the mixed problem

$$u_{tt} = c^2 u_{xx}, \text{ when } x > 0, t > 0;\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \text{ when } x > 0,$$

$$u_t(0,t) + a u_x(0,t) = h(t) \text{ when } t > 0,$$

where a denotes a constant.

8.3. Consider the system

$$u_t + Bu_x = Cu + D,$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} \sin^2 x \sin t - \sin x \cos x & \cos x \sin x \sin t + \sin^2 x \\ \cos x \sin x \sin t - \cos^2 x & \cos^2 x \sin t + \cos x \sin x \end{pmatrix},$$

and where C and D are matrices whose entries are smooth functions of x and t. Determine the eigenvalues for the system at every point $(x,t) \in \mathbf{R}^2$. Determine for which points $(x,t) \in \mathbf{R}^2$ the system is hyperbolic.

8.4. Rewrite the equation

$$u_{tt} = (1 + u_x)^2 u_{xx}$$

as a first-order system for $v = (u_x, u_t)^{\mathrm{T}}$. At which points $(x, t, z_1, z_2) \in \mathbf{R}^4$ is it hyperbolic?

8.5. Determine for which points $(x,t) \in \mathbf{R}^2$ the system

$$\begin{aligned} u_t + v_x &= v + w \\ v_t + u_x &= w \\ w_t + w_x \sin x &= u \end{aligned}$$

is hyperbolic.

9.1. Let b be a continuous function on the real axis such that $b(s)s \geqslant 0$ for all s, and consider solutions that are defined and continuous for $0 \leqslant x \leqslant \pi$, $0 \leqslant t$ to the problem

$$u_{tt} = c^2 u_{xx} - b(u_t), \qquad 0 < x < \pi, \quad 0 < t;$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = u(\pi,t) = 0.$$

Let $E(t)$ be the energy integral

$$E(t) = \tfrac{1}{2} \int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx.$$

Prove that E is decreasing. What does the term $b(u_t)$ signify in the equation? Solve the problem in the special case $b(s) = as$, where a is a constant satisfying $0 \leqslant a < 2c$ and with $f(x) = 0$, $g(x) = \sin mx$, $m \in \mathbf{N}$, $m > 0$.

9.2. Let φ be a real-valued test function on $\mathbf{R}^n$. Prove (using, e.g., the Fourier transformation) the following statements concerning the Laplacian Δ.

a) $\int \varphi \Delta \varphi dx \leqslant 0$.

b) There is a constant C_1 such that

$$\int \left| \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \right|^2 dx \leqslant C_1 \int (\Delta\varphi)^2 dx.$$

c) If $n \leqslant 3$, then there is a constant C_2 such that

$$\varphi(0)^2 \leqslant C_2 \int (\varphi^2 + (\Delta\varphi)^2) dx.$$

d) Determine a possible value for the constant C_2 when $n = 3$.

9.3. Determine a radial fundamental solution to the operator Δ^2 in $\mathbf{R}^n$ when $n \geqslant 3$.

9.4. Let (r, θ) be polar coordinates in the plane. Solve the Dirichlet problem

$$\Delta u = r \text{ for } r < 1; \qquad u = \sin\theta + \cos^2\theta \text{ for } r = 1.$$

Try different methods if you like.

9.5. Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} = 0; \qquad u(x,x) = 1, \quad \frac{\partial u}{\partial y}(x,x) = x^2.$$

9.6. A rod of infinite length has a temperature at time $t = 0$ which is given by the function e^{-x^2}. Heat conduction is assumed to occur according to the equation $u_{xx} = u_t$. Calculate the temperature of the rod at an arbitrary time $t > 0$. Show that the temperature at the point $x = 1$ on the rod first increases to a maximum value and then decreases.

10.1. Let $f \in C^2(\mathbf{R}^n)$ have the properties: $f(x) > 0$ when $\|x\| < 1$ and $f(x) = 0$ when $\|x\| \geqslant 1$. Determine in each of the three cases a, b, and c below the set $A_t = \{x \in \mathbf{R}^n;\ u(x,t) \neq 0\}$ for all $t > 0$.

a) Let $n = 1$ and let $u \in C^2(\mathbf{R} \times [0, +\infty[)$ be the solution to

$$u_t - u_{xx} = 0,\ x \in \mathbf{R},\ t > 0; \quad u(x,0) = f(x),\ x \in \mathbf{R}.$$

b) Let $n = 2$ and let $u \in C^2(\mathbf{R}^2 \times [0, +\infty[)$ be the solution to

$$u_{tt} - \Delta_x u = 0,\ x \in \mathbf{R}^2,\ t > 0; \qquad u(x,0) = 0 \text{ and } u_t(x,0) = f(x),\ x \in \mathbf{R}^2.$$

c) Let $n = 3$ and let $u \in C^2(\mathbf{R}^3 \times [0, +\infty[)$ be the solution to

$$u_{tt} - \Delta_x u = 0,\ x \in \mathbf{R}^3,\ t > 0; \qquad u(x,0) = 0 \text{ and } u_t(x,0) = f(x),\ x \in \mathbf{R}^3.$$

10.2. Let $P(\zeta)$ be a polynomial of n complex variables $(\zeta_1, ..., \zeta_n)$ of degree $m \geqslant n+1$ and with the properties that $|P(i\xi)| \geqslant (1 + \|\xi\|)^m$ when ξ is real. Then

$$E(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{P(i\xi)} d\xi$$

is a well-defined function, for the integral converges. Prove that E is a fundamental solution to the differential operator $P(D)$ which is obtained by substituting $\partial/\partial x_j$ for the variables ζ_j. Prove the formula

$$E(x) = (-1)^k \|x\|^{-2k} (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \Delta^k \left(\frac{1}{P(i\xi)} \right) d\xi$$

for every $k = 1, 2, \ldots$. From this formula we can deduce that E is a C^∞ function outside the origin, for $\|x\|^{2k} E$ can be differentiated quite a few times, depending on the fact that $\Delta^k(1/P(i\xi))$ decreases rather rapidly.

10.3. Let $b\colon \mathbf{R} \to \mathbf{R}$ be an increasing continuous function with $b(0) = 0$ and let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous, bounded and $\geqslant 0$. The initial-value problem

$$u_t - \Delta_x u + b(u) = 0, \qquad u(x,0) = f(x),$$

then has exactly one bounded solution and it is $\geqslant 0$. Moreover, if we let u and v be the solutions that belong to the initial values f and g, respectively, then $0 \leqslant f \leqslant g$ implies that $0 \leqslant u \leqslant v$. Prove that if the integral

$$(\otimes) \qquad \int_0^1 \frac{ds}{b(s)}$$

converges, then $u(x,t)$ will be zero for every bounded function f when t is sufficiently large. Prove that, conversely, if the integral $(\otimes)$ diverges, there exist functions $f > 0$ such that $u(x,t)$ never becomes zero for large t. *Hint:* As a starter, study solutions which depend on t only.