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Excercises
Ill-Posed Problems
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illposedexercises

1. Given an arbitrary index set J we define

$$l^2(J) = \left\{ (x_j)_{j \in J} \in \mathbf{R}^J ; \textstyle\sum_{j\in J} x_j^2 < +\infty \right\}.$$

It is a Hilbert space with the inner product

$$(x \,|\, y) = \sum_{j \in J} x_j y_j, \qquad x \in l^2(J),$$

and norm

$$\|x\|_2 = \sqrt{(x \,|\, x)}, \qquad x \in l^2(J).$$

(For each $x \in l^2(J)$, all coordinates x_j must vanish except for denumerably many indices j, so the sum has the usual sense.) Actually any Hilbert space is isomorphic to $l^2(J)$ for some J. We shall write l^2 for $l^2(\mathbf{N})$.

Consider now the following operators $S_k \colon \mathbf{R}^{\mathbf{N}} \to \mathbf{R}^{\mathbf{N}}$.

(a) $S_1(x)_j = x_{j+1}$, $j \in \mathbf{N}$, $x \in \mathbf{R}^{\mathbf{N}}$;

(b) $S_2(x)_j = x_{j-1}$, $j \geqslant 1$; $S_2(x)_0 = 0$, $x \in \mathbf{R}^{\mathbf{N}}$;

(c) $S_3(x)_j = \lambda_j x_j$, $j \in \mathbf{N}$, $x \in \mathbf{R}^{\mathbf{N}}$ for some real numbers λ_j.

Prove that S_1 and S_2 define continuous operators $T_1, T_2 \colon l^2 \to l^2$ by restriction. Prove that S_3 maps l^2 into l^2 if and only if the sequence $(\lambda_j)_j$ is bounded, and that its restriction T_3 to l^2 is continuous for the l^2 norm if this is the case.

Now consider the inverse problem of finding x when y is given. Show that the first two problems are ill-posed when $y \in l^2$ and $x \in l^2$. Prove that the third is well-posed if and only if the sequence (λ_j) is bounded away from zero. Does it make any difference here whether we use Hadamard's classical 1902 definition, which does not mention continuity, or the modern one, which includes continuous dependence on the data?

2. Consider the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$u(x, y, 0) = u(x, y, \pi) = 0; \quad u(0, y, t) = f(y, t), \quad u'_x(0, y, t) = g(y, t).$$

(The data are prescribed on the timelike manifold $x = 0$.) Verify that the functions

$$u_n(x, y, t) = a_n e^{nx} \sin(\sqrt{5} n y) \sin(2nt), \qquad n \in \mathbf{N},$$

solve the problem for certain data f_n, g_n. Define reasonable norms on some spaces containing the f_n, g_n and u_n for which the problem is ill-posed. Is it at all possible to define reasonable norms so that it is well-posed?

3. (a) Let E be a vector space and $A\colon E \to E$ a linear mapping. Define $E_\lambda = \ker(A - \lambda I)$, where I is the identity mapping and λ a scalar (real or complex). A number λ is said to be an *eigenvalue* of A if E_λ is nonzero, in other words if $A - \lambda I$ is not injective. If λ is an eigenvalue, the subspace E_λ is called the *eigenspace* belonging to λ, and a nonzero element of E_λ is called an *eigenvector*.

Example. Let $E = l^2 = l^2(\mathbf{N})$ and define A by the formula $A(x)_j = a_j x_j$, $j \in \mathbf{N}$. Then λ is an eigenvalue if and only if there is an index j such that $\lambda = a_j$, and the corresponding eigenspace is $\sum_j (E_{a_j}; a_j = \lambda)$. In this case, the sum of all eigenspaces is equal to the whole space.

Example. Let now $E = L^2([0,1])$, the space of all equivalence classes of square-integrable functions on the interval $[0,1]$. Define an operator A by declaring $A(x)(t) = g(t)x(t)$ for some bounded and measurable function g. Then

$$E_\lambda = \{x; x(t) = 0 \text{ for almost all } t \text{ such that } g(t) \neq \lambda\}.$$

Therefore there are no eigenvalues if $g(t) = t$ for example. On the other hand, if $g(t) = (t-a)^+ + b$, then $E_\lambda = \{0\}$ when $\lambda \neq b$, and

$$E_b = \{x; x(t) = 0 \text{ for almost all } t \in [a,1]\},$$

which is of infinite dimension when $0 < a \leqslant 1$. The sum of all eigenspaces can therefore be zero, but also the whole space.

(b) Let now $A\colon l^2 \to l^2$ be defined by $A(x) = \sum a_j x_j e_j$, where e_j are the usual basis vectors, and x_j the coordinates of x. The numbers a_j are the eigenvalues of A. We know that A is continuous if and only if $\sup_j |a_j|$ is finite. Prove that A is compact if and only if $a_j \to 0$ as $j \to +\infty$.

(c) With A defined as in (b) and assumed to be continuous and injective, let us define another mapping $S_\alpha\colon l^2 \to l^2$ by $S_\alpha(x) = \sum s_{\alpha,j} x_j e_j$, where α is a positive parameter. We know that S_α is continuous if and only if

$$\sup_j |s_{\alpha,j}| < +\infty. \tag{3.1}$$

Prove that $S_\alpha(A(x))$ tends to x for all x as $\alpha \to 0$ (in other words, the family $(S_\alpha)_\alpha$ is a regularization for A) if (3.1) holds for all $\alpha > 0$ and also

$$\text{for all } j \in \mathbf{N}, a_j s_{\alpha,j} \to 1 \text{ as } \alpha \to 0. \tag{3.2}$$

(d) Verify that the following choices of $s_{\alpha,j}$ are good for (c):

(3.3) $s_{\alpha,j} = 1/a_j$, $j = 0, 1, 2, ..., \lfloor 1/\alpha \rfloor$; $s_{\alpha,j} = 0$, $j > \lfloor 1/\alpha \rfloor$.

(3.4) $s_{\alpha,j} = 1/(\alpha + a_j)$, assuming all a_j to be positive.

(3.5) $s_{\alpha,j} = \min(1/\alpha, 1/a_j)$, $j \in \mathbf{N}$, assuming again all a_j to be positive.

How can we modify (3.4) and (3.5) if we only know that the a_j are nonzero?

(e) Prove that $\|S_\alpha\|$ is $\left|a_{\lfloor 1/\alpha \rfloor}\right|^{-1}$ in case (3.3), assuming the sequence (a_j) to be decreasing; equal to $(\alpha + \inf a_j)^{-1}$ in case (3.4); and equal to $(\max(\alpha, \inf a_j))^{-1}$ in case (3.5).

(f) Define now $R_n = \sum_0^n (I - A)^k$. Investigate under which conditions on the a_j we have, for all $x \in l^2$, $R_n(A(x)) \to x$ as $n \to +\infty$.

4. Consider a Volterra integral equation of the second kind:

$$u(x) - \int_0^x K(x,y)u(y)dy = f(x), \qquad x \in I = [0,1].$$

We define an operator A by

$$A(u)(x) = \int_0^x K(x,y)u(y)dy, \qquad x \in I,$$

so that the equation can be written $(I - A)u = f$. Find suitable spaces for f and u and suitable hypotheses on K under which a solution is given by the Neumann series

$$u = \sum_0^\infty A^k f.$$

5. Let G be (a model of) the Greenland ice, defined as $G = \{(t,z) \in \mathbf{R}^2; t \leqslant 0, z \geqslant 0\}$, where t is the time and z is the depth, defined to be positive below the ice surface. (In a more refined model, we would have restricted the depth to $0 \leqslant z \leqslant 3028.6\,\mathrm{m}$ and would also have considered the terrestrial heat flow from the underlying earth, but in this first study we simplify and consider $0 \leqslant z < +\infty$.) We consider *temperatures,* i.e., complex-valued continuous functions u on G which are of class C^2 and satisfy the heat equation $u_t = \kappa u_{zz}$ in the interior of G (for ice at -4°C the constant κ has the value $1.04 \cdot 10^{-6}\,\mathrm{m}^2\mathrm{s}^{-1}$). For each temperature there is a function $h(t) = u(t,0)$, $t \leqslant 0$, describing the temperature on the surface of the ice up to the present ($t = 0$), and a function $v(z) = u(0,z)$, $z \geqslant 0$, describing the present temperature in a hole in the ice.[1] We are interested in the possible operators $h \mapsto v$ (determining the present temperature at all depths from the surface temperature in the past, called the *direct problem*) and $v \mapsto h$ (determining the temperature in the past from the present temperature in the hole, called the *inverse problem*).

(a) Show by examples that it is not possible to determine uniquely the present temperature v from the surface temperature h in the past: there are many temperatures u such that $h(t) = u(t,0) = 0$ for all $t \leqslant 0$.

[1] Mnemonic trick: h for horizontal, v for vertical.

(b) Prove that if we assume in addition that u is bounded in G, then v is uniquely determined by h. Determine under which hypotheses the direct problem $h \mapsto v$ is well posed.

(c) Can you weaken the hypothesis in (b) that u be bounded and still conclude as in (b)?

(d) Consider simple functions $u(t,z) = e^{At+Bz}$, $(t,z) \in G$, $A, B \in \mathbf{C}$. Determine the exact conditions under which such a function is a temperature.

(e) Consider now $u(t,z) = e^{i\alpha t} e^{-\beta z + i\gamma z}$, $\alpha \in \mathbf{R}$, $\beta \geqslant 0$, $\gamma \in \mathbf{R}$, yielding $h(t) = e^{i\alpha t}$ and $v(z) = e^{(-\beta + i\gamma)z}$. Determine the necessary and sufficient relations between α, β, and γ for u to be a temperature.

(f) Perform a synthesis of the simple waves considered in (e): let h be a finite sum of the simple solutions we have found, say

$$h(t) = \sum A_k e^{i\alpha_k t}, \qquad t \in \mathbf{R},$$

where the α_k are real numbers such that $\alpha_j \neq \alpha_k$ for $j \neq k$. Or, a little more generally, consider

$$h(t) = \int_{\mathbf{R}} e^{i\alpha t} d\mu(\alpha), \qquad t \leqslant 0,$$

and a corresponding representation of v. Which are the conclusions for the well-posedness or ill-posedness of the problems $h \mapsto v$ and $v \mapsto h$? (To answer the question properly, you will have to introduce topologies on suitable spaces of functions.)

6. Consider the functional (energy) defined by David Mumford and Jayant Shah (1989),

$$E(u,K) = \lambda \int_{\Omega \smallsetminus K} (\operatorname{grad} u)^2 dx + \mu \mathcal{H}^{n-1}(K) + \nu \int_{\Omega} |u-g|^2 dx,$$

where λ, μ and ν are positive constants, Ω an open set in $\mathbf{R}^n$, $\mathcal{H}^{n-1}$ denotes Hausdorff measure in dimension $n-1$, and g is a given function. The variables are u and K: K a closed set in Ω and $u \in C^1(\Omega \smallsetminus K)$.

Let us study the appearance of discontinuities, i.e., the necessity to allow for a non-empty K in the formula to get close to the infimum.

Let us choose $n = 1$ and $\Omega =]-1,1[$, $\lambda = \nu = 1$, and $g(x) = \arctan ax$, $x \in \Omega$. Thus $\mathcal{H}^{n-1}(K) = \mathcal{H}^0(K) = \operatorname{card}(K)$. Let μ and a still be parameters to be fixed later.

Prove that there exists a constant $C > 0$ such that $E(u, \emptyset) \geqslant C$ for all u, all $\mu > 0$, and all $a \geqslant 1$.

Prove that

$$\inf_u E(u, \{0\}) \leqslant E(v, \{0\}) = \mu + \int_\Omega |v-g|^2 dx = \mu + \gamma_a,$$

where γ_a tends to zero as $a \to \infty$. Here v is chosen as a suitable constant in $]-1,0[$ and as another constant in $]0,1[$.

Thus if we now choose $\mu < C$, we will have $E(u,\emptyset) \geqslant C$ for all u while $\inf_u E(u,\{0\}) \leqslant E(v,\{0\}) \leqslant \mu + \gamma_a < C$ for large a. This shows that competing functions yielding a value close to the infimum must have a discontinuity when a is large—although perhaps not at the origin.

Continue the investigation in showing that some discontinuity of the competing functions u must converge to $\{0\}$ when $E(u,K)$ tends to its infimum. Hence any minimizing function must have a discontinuity at the origin when a is large, although g is C^∞ smooth.

7. Let us now consider a discrete variant of the Mumford–Shah functional; cf. Geman and Geman (1984). We define a norm in $\mathbf{R}^n$, a modified l^2 norm, by

$$\|u\|^2 = \frac{1}{n}\sum_1^n u_j^2, \qquad u \in \mathbf{R}^n,$$

and a mean value

$$M(u) = \frac{1}{n}\sum_1^n u_j, \qquad u \in \mathbf{R}^n.$$

Given g, the energy is

$$E(u) = \frac{1}{n-1}\sum_1^{n-1} W(u_{j+1}-u_j) + \|u-g\|^2, \qquad u \in \mathbf{R}^n,$$

where W is the function

$$W(t) = \min(\alpha t^2, \beta), \qquad t \in \mathbf{R},$$

α and β being positive constants. The role of the ceiling β is to allow for jumps: large jumps are counted, but not more than by a certain amount even if they are very large; this is exactly the role of the term $\mathcal{H}^0(K)$ in the previous problem.

We note that, taking u as the constant $M(g)$,

$$\inf_u E(u) \leqslant E(M(g)) = \|g\|^2 - M(g)^2,$$

and that, taking $u = g$,

$$\inf_u E(u) \leqslant E(g) = \frac{1}{n-1}\sum_1^{n-1} W(g_{j+1}-g_j).$$

Thus the infimum can be estimated from above by the minimum of three numbers,

$$\inf_u E(u) \leqslant \min\left(\|g\|^2 - M(g)^2, \beta, \frac{\alpha}{n-1}\sum_1^{n-1}(g_{j+1}-g_j)^2\right).$$

To illustrate the discontinuity of the minimizing function as a function of g, take now $n = 2$ and g a function with $M(g) = 0$.

(a) Show that it is enough to consider u with $M(u) = 0$ to get close to the infimum.

(b) Find the minimizing function u depending on g and show that it does not depend continuously on g.

In fact, when $g_1^2 > \beta(1 + 1/4\alpha)$, then the infimum is attained when $u_1 = g_1$, wheras when $g_1^2 < \beta(1+1/4\alpha)$, then the infimum is attained when $u_1 = g_1/(1+4\alpha) \neq g_1$. The value of the infimum is $\inf_u E(u) = \min\left(\beta, g_1^2/(1 + 1/4\alpha)\right)$.

This shows that when g varies a lot, i.e., $\max g_j - \min g_j = |g_1 - g_2| = 2|g_1|$ is large, it pays to let $u = g$ and let the whole energy occur in the first term of E (analogous to the jump discontinuity in problem 6), whereas when g does not vary so much, u can differ from g and yield some energy as measured by the second term.

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