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ÖVNINGAR, Blad 4
Transformer för beräkningar
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4.1. Let $u, v \in l^2(\mathbf{Z}_8)$ be the vectors

$$u = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0, 0, 0, 0), \quad v = (1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0, 0, 0, 0).$$

(The first-stage Haar basis.)

(a) Prove that the vectors $R_{2k}u, R_{2k}v$, $k = 0, 1, 2, 3$, form an orthonormal system in $l^2(\mathbf{Z}_8)$. (3)

(b) Define

$$P(z) = \sum_{k=0}^{3} \langle z, R_{2k}u \rangle R_{2k}u, \qquad Q(z) = \sum_{k=0}^{3} \langle z, R_{2k}v \rangle R_{2k}v.$$

Calculate $P(z)$ and $Q(z)$ when $z = (1, 2, 4, 6, 10, 12, 10, 8)$. Tenta 2000 05 22:5

4.2. Suggest an $O(N \log N)$ algorithm for computing matrix-vector multiplication, assuming the matrix to be Toeplitz. (Recall that the elements of a Toeplitz matrix are constant along diagonals. A circulant matrix is Toeplitz but not necessarily conversely.) Tenta 2000 05 22:6

4.3. Define $f\colon \mathbf{R}^2 \to \mathbf{R}$ by $f(x) = 1 - \|x\|_2^2$ when $\|x\|_2 < 1$, $f(x) = 0$ when $\|x\|_2 \geqslant 1$. Calculate the Radon transform $\varphi = \mathcal{R}f$ of $f$, i.e., calculate $\varphi(\omega, p)$ for $\omega = (\cos\theta, \sin\theta) \in S^1$, $p \in \mathbf{R}$. Tenta 2000 05 22:7

4.4. For any $z \in l^\infty(\mathbf{Z})$ let us denote by $K(z)$ the smallest interval $I \subset \mathbf{Z}$ such that $z(j) = 0$ for all $j \notin I$. ($K(0) = \varnothing$.) Prove that if $K(z)$ and $K(w)$ are of finite length, then

$$K(z * w) = K(z) + K(w).$$

(An easy form of Titchmarsh' theorem on supports.) Show by an example that this may not be true if $K(z)$ or $K(w)$ is infinitely long. But it is true if $K(z)$ and $K(w)$ are contained in the set of nonnegative integers.

4.5. Prove that $\hat{z}$ is a continuous function, $\hat{z} \in C(\mathbf{R})$, if $z \in l^1(\mathbf{Z})$. Here $\hat{z}(t) = \sum z(k)e^{ikt}$, $t \in \mathbf{R}$.

4.6. Prove that if $f \in C^2(\mathbf{R})$ is periodic with period $2\pi$, then $\check{f} \in l^1(\mathbf{Z})$. Here $\check{f}(n) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(t)e^{-int}dt$, $t \in \mathbf{R}$.

4.7. Prove that the convolution equation $a * z = b$ can be solved with $z \in l^1(\mathbf{Z})$ if $\hat{a}, \hat{b} \in C^2(\mathbf{R})$ and have period $2\pi$, and if in addition we assume that there exists an $\varepsilon > 0$ such that $|\hat{a}(t)| \geqslant \varepsilon$ for all $t$ such that $\hat{b}(t) \neq 0$.

4.8. Prove that the convolution equation $z * z = \delta$ has exactly two solutions $z \in l^1(\mathbf{Z})$, but infinitely many solutions $z \in l^2(\mathbf{Z})$.

4.9. When does the FRFT-algorithm perform better than the usual mixed-radix implementation of FFT? Motivate your answer.

4.10. Consider a filterbank transform of $z \in Z_N$, where $N = 2^p$ and the coarsest level in the transform has $N_0 = 2^q$ points, independent of $N$. Assume that the filters have the same number of non-zero elements on all levels. Show that the transform can be computed in $\mathrm{O}(N)$ arithmetic operations.

4.11. Let $Q$ be a matrix representing some fast transform. Assume that $A$ is diagonalised by $Q$. Show that the eigenvalues of $A$ are given by

$$\lambda = mQD^{-1},$$

where $m$ is a row of $A$ and the corresponding row in $Q$ are the diagonal entries of the diagonal matrix $D$.

4.12. Suggest a fast algorithm for solving a system of equations with $A$ as coefficient matrix.