On Maximal Balls in Three Volume Grids

Gunilla Borgefors1*, Robin Strand2

1: Swedish University of Agricultural Sciences, 2: Uppsala University Centre for Image Analysis, Box 337, SE-75105, Uppsala, Sweden

E-mail: {gunilla, robin}@cb.uu.se

Abstract: A volume image can be digitized in different grids, not only the cubic one. The fcc and bcc grids have many advantages, as they are more dense than the cubic one. The set of maximal balls in a shape in a volume image is a compact but complete description of the shape. The original set, identified by rules dependent on the metric used, can be further reduced, by observing that some balls are covered by groups of other balls. The set of maximal balls can, for example, be used for compression, manipulation and as anchor points for topologically correct medial representations.

Keywords: Volume image, lattice, metric, distance transform, medial axis, skeleton

1 MAXIMAL BALLS

The concept of “medial axis” or “medial representation” or “skeleton” in digital images starts with a paper of Blum 1967 [2]. Here, we will use medial representation (MR). The underlying idea is to simplify the representation of a shape $S$ by reducing its dimension. Objects in $\mathbb{R}^2$ are represented by a set of curves that are easier to analyze and manipulate than the objects themselves.

Transferring Blum’s “grass-fire” medial axis from the continuous plane to the digital world is not straightforward. There are (at least) three definitions of the MR of $S$ in $\mathbb{R}^2$ (that can all be extended to higher dimensions):

1. Start a grassfire at the border of $S$. The quenching points of the fire, i.e., the points that are equally distant from at least two different points on the border, is $\text{MR}_1$.

2. Consider a disc $D$ that is completely inside the shape. If there is no other such disc that completely covers $D$, then $D$ is a maximal disc. The centres of the maximal discs is $\text{MR}_2$.

3. The centres of all discs $D$ that touch the border of $S$ at more than one point is $\text{MR}_3$.

In $\mathbb{R}^2$ these three definitions give exactly the same MR. This MR is set of curves that have the same topology as the original shape, i.e., it has the same number of components and the same number of holes. But this is not the case in $\mathbb{Z}^2$. The sets $\text{MR}_1$, $\text{MR}_2$, and $\text{MR}_3$ are different. The sets also depend on the metric used to compute discs and distances in the digital world. Blum assumes the Euclidean metric, but this is not always the best choice in the digital case. These interpretation problems are the reason that there are so many conflicting definitions and so many different approaches to MR computation in image analysis.

If the emphasis is on representation, then it should be possible to recreate $S$ from its MR. This is not the case for $\text{MR}_3$. If, e.g., a shape is an even number of pixels wide, then $\text{MR}_3$ is empty. $\text{MR}_2$, with the radii of the maximal discs preserved, gives reproducibility, as the union of maximal discs equals $S$ in both $\mathbb{R}^n$ and $\mathbb{Z}^n$, independently on the metric used. $\text{MR}_1$ contains additional pixels, not needed for reproducibility. Therefore, $\text{MR}_2$ is a good definition to start from in $\mathbb{Z}^n$.

In $\mathbb{R}^2$, MR is topologically correct. None of the three definitions gives a topologically correct and thin MR in $\mathbb{Z}^2$. The sets can be disconnected and they are two pixels thick where the shape is evenly wide. Further processing is necessary to produce thin, connected digital curves.

2 VOLUME GRIDS

The two-dimensional hexagonal grid has advantages over the traditionally used square grid. For example, less samples are needed to get the same reconstruction quality, it is less rotational dependent, and each picture element has only one type of neighbour, which simplifies many algorithms. The corresponding three-dimensional grids are the face-centred cubic (fcc) grid and the body-centred cubic (bcc) grid. The fcc grid is spanned by, e.g., the vectors $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$. The voxels (Voronoi regions) on the fcc grid are rhombic dodecahedra. The bcc grid is spanned by, e.g., the
vectors \((1,1,1), (1,-1,1),\) and \((1,1,-1)\). The voxels (Voronoï regions) are truncated octahedra. See also [22].

Because of the high packing densities, only 0.770 and 0.707 of the number of samples in the cubic grid are needed for the fcc and bcc grids, respectively, to fulfill the Shannon sampling theorem. It follows (see, e.g., [13, 22] for details) that by using a bcc grid, almost 30\% less samples can be used without affecting the reconstruction/representation quality.

Most volume image acquisition techniques can be modified to acquire images directly on the fcc or bcc grids. In [22], the benefit of using these grids in computed tomography with reconstruction methods like the filtered back-projection, the direct Fourier method, and the algebraic reconstruction technique is discussed.

The fcc grid \(F\) and the bcc grid \(B\) are defined as:

\[
F = \{(x,y,z) \in \mathbb{Z}^3 : x + y + z \equiv 0 \pmod{2}\},
\]

\[
B = \{(x,y,z) \in \mathbb{Z}^3 : x \equiv y \equiv z \pmod{2}\}.
\]

Their Voronoï regions (voxels) are shown in Fig. 1.

![Fig. 1: A voxel (white) and its neighbours. The light grey, dark grey, and black voxels are 1-, 2-, and 3-neighbours, respectively. From left to right: \(B\), \(F\), and \(Z^3\).](image)

Two distinct points \((x_1,y_1,z_1),(x_2,y_2,z_2)\) in either the fcc or bcc grid are neighbours of order \(r\), \(1 \leq r \leq 2\), if

1. \(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| \leq 3\) and
2. \(\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} \leq r\).

In the cubic grid, the points \((x_1,y_1,z_1),(x_2,y_2,z_2)\) are neighbours of order \(r\), \(1 \leq r \leq 3\), if

1. \(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| \leq r\) and
2. \(\max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} \leq 1\).

The neighbours in the three grids are found in Fig. 1.

3 METRICS

Three types of metrics are usually considered in digital images, [4].

- Path-generated metrics, where the distance between two voxels equals the number of voxels in the path between them.
- Weighted metrics, where different types of neighbours in the path are given different weights.
- The (squared) Euclidean distance.

The path-generated metrics are easiest to compute and, more important, easiest to use in various algorithms, whereas the Euclidean metric is least direction dependent but usually rather complex to use. Weighted metrics are in between and often a good compromise.

Here, we consider two metrics for each grid, path-generated using only 1-neighbours (corresponding to the city block distance in \(\mathbb{Z}^2\)) and integer weighted metrics.

Let the weights to 1- and 2-neighbours be \(\alpha\) and \(\beta\), respectively. On the cubic grid, 3-neighbours are also used; we denote their weight by \(\gamma\).

Weight optimizations for these grids can be found in [4, 26, 23, 12, 21, 24]. We use the following integer approximations: \((\alpha, \beta) = (2,3)\) for the fcc grid, \((\alpha, \beta) = (4,5)\) for the bcc grid, and \((\alpha, \beta, \gamma) = (3,4,5)\) for the cubic grid.

The following expressions for the distance transform values are found in [4, 22]. In \(F\):

\[
d(0, (x,y,z)) = \begin{cases} 
\alpha x + \beta y + \gamma z & \text{if } x \leq y + z \\
(y + z)\alpha + \frac{x - y}{2}\beta & \text{otherwise.}
\end{cases}
\]

In \(B\):

\[
d(0, (x,y,z)) = \alpha x + \frac{x - y}{2}\beta.
\]

In \(\mathbb{Z}^3\):

\[
d(0, (x,y,z)) = (x - y)\alpha + (y - z)\beta + z\gamma.
\]
The path-generated metrics can also be considered as weighted metrics, with $\alpha = 1, \beta = 2$ for both $\mathbb{F}$ and $\mathbb{B}$; and with $\alpha = 1, \beta = 2, \gamma = 3$ for $\mathbb{Z}^3$. Which these weights, all expressions and theory valid for weighted metrics are valid also for the path-generated metrics, as the weights fulfil the restrictions in (1)-(3).

In the distance transform of $S$, each voxel has the value of the distance from the nearest background voxel in the chosen metric. Distance transforms, including the Euclidean one, can be computed using various algorithms. For the cubic grids see [4, 14] and for fcc and bcc see [12, 25, 22].

4 FINDING MAXIMAL BALLS

To find the maximal balls, the set $M_{\mathcal{R}_S}$, of $S$, first compute its distance transform. A central voxel $v_0$ is compared to the values of its neighbours, denoted $v_1, v_2, v_3$ for 1-neighbours, 2-neighbours, and 3-neighbours, respectively.

If the path-generated metric is used, then the MR consists of those voxels $v_0$ that have a higher or equal distance value than all its 1-neighbours, i.e., $v_0 \geq v_1$.

If a weighted metric is used, the principle is the same, except that the smallest weight between voxels must be added before the comparison is made. A voxel $v_0$ is a centre of a maximal disc and thus belongs to MR if

$$v_0 + \alpha > v_1 \quad \text{and}$$
$$v_0 + \beta > v_2 \quad \text{and}$$
$$v_0 + \gamma > v_3 \quad \text{for } \mathbb{Z}^3. \quad (4)$$

These rules are not valid for small distance values [22, 1, 3, 15], but by replacing some values in the distance transform with other values (4) can be used for all voxels $v_0$. The replacements for our weighted distance transforms are:

- For $\mathbb{F}$, $\alpha = 2, \beta = 3$,
  set all voxels with value 2 to 1,

- For $\mathbb{B}$, $\alpha = 4, \beta = 5$,
  set all voxels with value 4 to 1,
  with value 8 to 6, and with value 12 to 11,

- For $\mathbb{Z}^3$, $\alpha = 3, \beta = 4, \gamma = 5$,
  set all voxels with value 3 to 1.

In general, for integer weighted metrics using 1-neighbours and 2-neighbours, each value should be replaced by the lowest value under it that cannot exist in the distance transform. In $\mathbb{F}$, with $\alpha = 2, \beta = 3$, all integers are possible except 1, so 2 should be replaced by 1.

If the Euclidean metric is used, then look-up tables must be used to find MR, see [10, 16] for the cubic grid.

The definition of a maximal ball is that it is covered by no other single ball inside the shape $S$. But a maximal ball may very well be completely covered by several other maximal balls. This means that the original set MR computed by (4) can be reduced by removing balls not necessary to reproduce the shape.

Coeurjolly and co-authors have proved that finding the minimal set of maximal discs for the Euclidean metric in $\mathbb{Z}^2$ is an NP-hard problem [11]. There is no reason to believe that this is not the general case. However, a good heuristic algorithm exists, [5]. It is not fast, but it is simple and does produce very good results, [11].

Start with the value zero in all voxels in $S$. For each voxel $v_{\text{MR}}$ in MR, add 1 to all voxels in the maximal disc associated with $v_{\text{MR}}$. This can, for example, be done by the reverse distance transform, [1, 8, 10], where the MR with its distance value act as a seed point. Repeat for all voxels in MR to produce a “covering map”.

When the covering map is complete, the elimination begins. For each voxel in MR, find the minimal value, $\text{MR}_{\text{min}}$ in the covering map for all voxels in its associated ball. If $\text{MR}_{\text{min}} > 1$ then the voxel can be removed from MR and all voxels in its ball should be reduced by one. The elimination starts with the balls with the smallest radii and ends with the maximal radius. The result is the reduced MR, rMR.

The shape $S$ can be completely recovered from rMR using the reverse distance transform.

A simple way of smoothing $S$ and further reducing rMR is to remove all maximal balls with small radii. This will remove small details of the recovered $S$, but keep the general shape.
5 EXAMPLES

As an example, we use the shape of the antibody IgG, a protein molecule, which consists of a body and two flapping “ears”. In Fig. 2 the shape is digitized in the three grids, using about the same number of voxels.

In Table 1 the number of voxels in the original shapes and the percentages remaining in the MR and rMR for the two metrics and the three grids are shown. A general observation is that reducing the set of maximal balls results in a much smaller set for the weighted metrics than for path-generated metrics.

The MR and rMR for the weighted metrics are shown in Figs. 3 and 4.

Table 1: Numbers of voxels in the different sets.

<table>
<thead>
<tr>
<th>Set</th>
<th>fcc</th>
<th>bcc</th>
<th>cubic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>5760</td>
<td>5899</td>
<td>5579</td>
</tr>
<tr>
<td>path metric</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>9.88%</td>
<td>8.34%</td>
<td>9.00%</td>
</tr>
<tr>
<td>rMR</td>
<td>7.93%</td>
<td>6.03%</td>
<td>8.57%</td>
</tr>
<tr>
<td>weighted metric</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>9.17%</td>
<td>10.14%</td>
<td>18.84%</td>
</tr>
<tr>
<td>rMR</td>
<td>4.32%</td>
<td>4.42%</td>
<td>4.46%</td>
</tr>
</tbody>
</table>

6 APPLICATIONS

In addition to compression of shapes, the set rMR computed here is often used as an anchor set to compute thin, topologically correct skeletons (or medial axes). Including these points ensures that the shape can be completely recovered from the skeleton. Points must be added to make the skeleton topologically correct, which is relatively easy in 2D. In 3D, the skeleton consists of surface patches and connecting the rMR into nice surfaces is not trivial.

A reversible skeleton cannot be thin everywhere, as shapes of even width will have two-point thick rMR. But skeletons are much easier to handle if they are thin, so curves and surfaces can be traced. Therefore, a final thinning that removes a few border elements from the recovered shape is usually performed.

Skeletons computed from MR or rMR in $\mathbb{Z}^2$ are found in [6, 28, 18, 7] and surface skeletons in $\mathbb{F}$ and $\mathbb{B}$ in [20].

References


Fig. 2: The antibody IgG in the cubic (5579 voxels), fcc (5899 voxels), and bcc (5760 voxels) grids.

Fig. 3: The sets of maximal balls for IgG (Fig. 2) using integer weighted metrics in the cubic, fcc, and bcc grids.

Fig. 4: The reduces sets of maximal balls for IgG (Fig. 2) using integer weighted metrics in the cubic, fcc, and bcc grids.