A tutorial on direction estimation and orientation representations

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Introduction

Problem 1
Assign orientation to each pixel using local information
- Strategies
- Uncertainty principles

Problem 2
Orientation for larger regions, circular data etc...
- Averaging
- Representation
- Orientation vs direction

Type 1 problems
Local estimation
- Any small patch
- Lines
- Edges

Type 2 problems
Semi local/global estimation
- Averaging
- Multiple orientations
- Symmetric representations

Local estimation of orientation

Taylor series
Jacobian/Gradient

One variable
\[ f(x) = f'(0) x + \frac{f''(x)}{2!} x^2 + \ldots \]

Several variables (dimensions)
Let \( f : \mathbb{R}^N \to \mathbb{R} \), then
\[ f(x) = f(0) + Df(0)^T x + \ldots \]
\[ Df(x) = \left( \frac{\partial f}{\partial x_1}(0), \frac{\partial f}{\partial x_2}(0), \ldots, \frac{\partial f}{\partial x_N}(0) \right) \]
and \( Df : \mathbb{R}^N \to \mathbb{R}^N \)
Gradient as a Least Squares Problem

\[
\begin{pmatrix}
\frac{1}{9} & -1 & 1 \\
\frac{1}{9} & -1 & 0 \\
\frac{1}{9} & -1 & -1 \\
\frac{1}{9} & 0 & 1 \\
\frac{1}{9} & 0 & 0 \\
\frac{1}{9} & 0 & -1 \\
\frac{1}{9} & 1 & 1 \\
\frac{1}{9} & 1 & 0 \\
\frac{1}{9} & 1 & -1 \\
\end{pmatrix}
\begin{pmatrix}
c_0 \\
dx \\
dy \\
\end{pmatrix} = \begin{pmatrix}
M(1,1) \\
M(2,1) \\
M(3,1) \\
M(1,2) \\
M(2,2) \\
M(3,2) \\
M(1,3) \\
M(2,3) \\
M(3,3) \\
\end{pmatrix}
\]

Finite differences
Directions from discretized derivatives I

- Very local, depends only on three pixels.
- Half pixel offset!
- Rotationally invariant? (I.e. do we get the same result if we rotate the image first, then calculate the gradient, and then rotate back?)

See fig (1,2) and (1,3)!

Least Squares and Projections

Say we have some data points \( \{y_i\} := y(x_i), i = 1, ..., N \) and a basis function \( \{b_i\} \). Now we want to find the \( c \) that minimises

\[
E(c) = ||cb(x) - y(x)||.
\]

In the least squares approach, we expand Eq. 1 as

\[
E(c) = \sum_{i=1}^{N} \left[ c^2 b_i^2 - 2c b_i y_i + y_i^2 \right].
\]

Least Squares and Projections

Derivation with respect to \( c \):

\[
\frac{d}{dc} E(c) = \sum [c b_i^2 - b_i y_i] = 0,
\]

gives

\[
c = \frac{\sum b_i y_i}{\sum b_i^2}.
\]

With the projection approach, the projection of \( y \) to \( b \) is expressed

\[
\text{Proj}_b y = b \left( \frac{y}{||b||^2} \right)
\]

so we identify

\[
c = \frac{(b, y)}{||b||^2} = \frac{b^T y}{\sum b_i^2}.
\]

Finite differences
Directions from discretized derivatives I

Second smallest filter: \([1.0,-1]/2\)

In matlab

\[
dx=\text{convn}(I, [1,0,1,0,1,...]);
\]

- Invariant to \([...1,0,1,0,1,...]\)
- Still a little too discrete?
- Symmetric
- Local

Least Squares and Projections

When the solution \( x \) to \( Ax = b \) can’t be found by matrix inverse (i.e. too low rank), we can find

\[
\arg \min_x ||Ax - b||^2
\]

i.e.

\[
x = (A^T A)^{-1} A^T y
\]
A function $y = g_{\partial}(x, d, \sigma)$ can be defined as follows:

1. **function** $y = g_{\partial}(x, d, \sigma)$
2. $w = 10 \times \text{ceil}(\sigma)$; $w = w + \text{mod}(w + 1, 2)$; % Filter length
3. $g = \text{fspecial}(\text{'gaussian'}, [w, 1], \sigma)$; % 1D Gaussian
4. $x = (-[w-1]/2:(w-1)/2)$;
5. $k0 = 1/\sqrt{2\pi\sigma^2}$; $k1 = 1/(2\sigma^2)$;
6. $dg = 2*k0*k1*x.*\exp(-k1*x.^2)$; % $d/dx$ Gaussian
7. if $d = 1$
8. $y = \text{convn}(x, \text{reshape}(dg, [w, 1]), \text{'same'})$;
9. $y = \text{convn}(y, \text{reshape}(g, [1, w]), \text{'same'})$;
10. end
11. if $d = 2$
12. $y = \text{convn}(x, \text{reshape}(g, [w, 1]), \text{'same'})$;
13. $y = \text{convn}(y, \text{reshape}(dg, [1, w]), \text{'same'})$;
14. end

See Koenderink and Lindeberg!

### A1: Sub pixel location of extremal points

Another example of Taylor expansion in image analysis. Sub pixel location of local extreme points is achieved by second order Taylor expansion using the $3^N$ closest points by:

$$D(x) = D + \frac{\partial D^T}{\partial x} x + \frac{1}{2} x^T \frac{\partial^2 D}{\partial x^2} x$$

$$\hat{x} = -\left(\frac{\partial^2 D}{\partial x^2}\right)^{-1} \frac{\partial D}{\partial x}$$

### A2: Location of edges

A one dimensional signal $P(x)$. We define a unit step edge located at $x = 0$ by

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

**Def 1**: An edge can be located where the first derivative of the signal has an extremal value (zero crossing of second derivative)

$$E_D = \{x : \frac{d^2}{dx^2} P(x) = 0\}.$$  

**Def 2**: The edge can be located where the signal obtains a specific value or level $c$, i.e. the set

$$E_l = \{x : P(x) = c\}.$$  

### A2: Location of edges

**Definitions**

**Step edges**: $P(x) = G_x \ast \theta(x)$, $E_D$ is the set of points that satisfy

$$0 = \frac{d^2}{dx^2} G_x \ast \theta(x) = \frac{d^2}{dx^2} \text{erf}_x(x) = G_x'(x).$$

which are

$$x = \pm \sqrt{\frac{2ab}{4b^2 - 2ab}} = \pm \sqrt{3\sigma}.$$  

Two detections, none at $x = 0$.  

**Lines**: $P = \delta(x) \approx \frac{1}{\epsilon} (\theta(x) - \theta(x + \epsilon))$ for a small $\epsilon$. $E_D$ is the set of points that satisfies

$$0 = \frac{d^2}{dx^2} \delta \ast G_x(x) = e^{-bx^2} (4x^2b^2 - 2ab),$$

which are

$$x = \pm \sqrt{\frac{2ab}{4b^2 - 2ab}} = \pm \sqrt{3\sigma}.$$  

Two detections, none at $x = 0$.  

**Equivalences**

- Rotational invariance $\Rightarrow$ round support
- Non ringing $\Rightarrow$ smooth radial profile
- $\Rightarrow$ Gaussian derivatives!

- Convolutions
- Projection on linear bases
- Least squares solutions

**Generalises to ND**
A2: Location of edges

Unit ridges $P(x) = \theta(x) - \theta(x - w)$, where $w > 0$ is the width.

$E_D$ contains the points that satisfy

$$0 = G'_s(x) - G'_s(x - w) = -2abxe^{-bx^2} + 2ab(x - w)e^{-b(x^2 - 2wx + w^2)},$$

(15)

or simplified

$$0 = 2abe^{-b x^2} \left\{ (x - w)e^{-b(w^2 - 2wx)} - x \right\}.$$  

(16)

which further can be reduced to

$$0 = x - (x - w)e^{-b(w^2 - 2wx)}.$$  

(17)

Neither $x = 0$ or $x = w$ are solutions.

For an ideal step edge,

$$E_I = \{ x : G_s * \theta(x) = \text{erf}_s(x) = c \},$$

(18)

and since $E_I = \{0\}$ is required, $c = 1/2$.

For lines, no, one or two edges will be detected since the condition is that

$$E_I = \{ x : G_s(x) = 1/2 \}.$$  

(19)

For finite ridges,

$$E_I = \{ x : \text{erf}_s(x) - \text{erf}_s(x - w) = 1/2 \}.$$  

(20)

none (or one) or two edges are detected.

Figure: Left: A unit ridge, smoothed, its second derivative (scaled) and the $1/2$ line. Right: NW: A ridge. NE: Smoothed with $\sigma/w = 0.7$. SW: Canny edge detection. SE: Pixels with intensity above 0.5.

GOP / Quadrature Filters

Phase invariance

$$F(u, v) = F(r, \theta) = R(r)T(\theta).$$  

(21)

$$R(\rho) = e^{-\frac{\rho^2}{4B^2 \ln 2 \ln 2}}(\rho/\rho_0)$$  

(22)

$$T_d(u) = \left\{ \begin{array}{ll} \langle u, d \rangle^2, & \langle u, d \rangle > 0, \\ 0, & \langle u, d \rangle \leq 0. \end{array} \right.$$  

(23)

radial part
angular part
combined
→ can also be achieved by averaging techniques.

Quadrature filter in the spatial domain
Intermediate summary:
- Gaussian derivatives to calculate gradients!
- Gradients vanish for some structures.
- Higher order constructions are needed. One such technique is the Hessian.
- Phase invariant filters are good.
- Sub pixel location of edges is not trivial (see Van Vleet)

Representing directions and orientations

Histograms

- Angular Histogram, $[0, 2\pi]$ is divided into eight bins
- Gradient directions $\theta = \text{atan2}(dx, dy)$.
- 16 spatial bins, 8 angular bins (128)
- Gaussian Weights
- 128-dimensional

Approaches from the SIFT family

SIFT
GLOH
SURF

Properties of Histograms

- Discretizations
- Number of bins
- Rotations do no commute
- Discontinuous at $2\pi = 0$
- Quantitative
- Tessellation in $\mathbb{R}^2$

Another representation?
1. Commuting rotations
2. Discontinuity-free
3. Perfect retrieval

Kernel Density Estimators (KDE)


"Given a sequence of independent identically distributed random variables $X_1, X_2, ..., X_n$, ... with common probability density function $f(x)$, how can one estimate $f(x)$?"

- Extensions to manifolds
- A standard approach, > 5000 citations.
The structure tensor constructed from the same angle $s = (\cos \theta_0, \sin \theta_0)$ is

$$S = s^T = \begin{pmatrix} \cos^2 \theta_0 & \sin \theta_0 \cos \theta_0 \\ \sin \theta_0 \cos \theta_0 & \sin^2 \theta_0 \end{pmatrix}.$$
The outer product \((\nabla I)^T \nabla I\)

The gradient of \(I\) is
\[
\nabla I = \left( \frac{\partial}{\partial \delta x_1}, \frac{\partial}{\partial \delta x_2}, \frac{\partial}{\partial \delta x_3} \right),
\]
so the structure of the outer product
\[
E := (\nabla I)^T \nabla I \approx \begin{pmatrix}
a^2 & ab & ac \\
ab & b^2 & bc \\
ac & bc & c^2
\end{pmatrix}. \tag{24}
\]

- \(E\) is Self-Adjoint since it is real and symmetric.
- The Spectral Theorem for real vector spaces then states that the eigenvectors to \(E\), \(v_i\), form an orthonormal (ON) basis.
- A shorter proof that the eigenvectors corresponding to distinct eigenvalues are ON. Assume that \(Ed = \delta\) and \(Ee = \epsilon\) then
\[
(\delta - \epsilon) \langle d, e \rangle = \langleTd, e\rangle - \langle d, T^* e \rangle = \langleTd, e\rangle - \langleTd, e\rangle = 0
\]
and since \(\delta - \epsilon \neq 0\), it hold that \(\langle d, e \rangle = 0\).

Using \(S_x\) as a quadratic form

Denote the eigenvalues to \(S_x\) as \(\lambda_i\) and the eigenvectors \(v_i\). Then the structure tensor maps vectors as
\[
\langle E w, w \rangle = \langle \lambda_1 \text{Proj}_{x_1} w + \lambda_2 \text{Proj}_{x_2} w + \lambda_3 \text{Proj}_{x_3} w, w \rangle
= \lambda_1 \langle w, v_1 \rangle v_1, w \rangle + \lambda_2 ...
= \lambda_1 \langle w, v_1 \rangle \langle w, v_1 \rangle + \lambda_2 ...
= \lambda_1 \cos^2 \theta_1 + \lambda_2 \cos^2 \theta_2 + \lambda_3 \cos^2 \theta_3
\]
Where the angles \(\theta_j\) is the angle between \(w\) and each eigenvector, \(v_j\).

The 2x2 eigenvalue problem

Eigenvalues
The eigenvalue problem \(\det Ax = \lambda x\) has the characteristic polynomial \((a - \lambda)(c - \lambda) - b^2 = 0\) when \(A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\) and the solutions \(\lambda = \frac{-b \pm \sqrt{b^2 - ac + \left(\frac{a - c}{2}\right)^2}}{2}\), equivalent to \(\lambda = \text{Tr}/2 \pm \sqrt{\left(\text{Tr}/2\right)^2 - D}\), where \(\text{Tr} = \text{Trace} A\) and \(D = \det A\).

Eigenvectors
If we set \(x_1 = 1\), we get \(x_2 = -b/(c - \lambda)\). When \(b \approx 0\), \(A\) is diagonal and \(x = (1, 0)^T\) when \(\lambda \approx a\) and \((0, 1)^T\) when \(\lambda \approx c\).

The symmetric eigenvalue problem

Introduction

1. The 3x3 eigenvalue problem i.e. to find \(x \in \mathbb{R}^3 - (0, 0, 0)\) and \(\lambda \in \mathbb{R}\) which satisfies \(Ax = \lambda x\) for \(A = A^T \in \mathbb{R}^{3 \times 3}\).
2. Multiple approaches possible.
3. Cardano’s solution to the characteristic equation \((\det(Ax - \lambda I)) = 0\) is not suited for numerical computations. (Demmel)
4. Jacobis method is the fastest?

A plane rotation matrix
\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]
has the properties \(R^{-1}(\theta) = R(-\theta)\). A \(2 \times 2\) real and symmetric matrix
\[
M = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}
\]
can be diagonalised with such rotation matrix so that
\[
R^{-1}M R = D. \tag{25}
\]
After the rotation, \(D\) and \(M\) are similar, i.e. have the same eigenvalues.
that makes $D$ diagonal is not explicitly needed:
\begin{align*}
\epsilon &= \frac{\alpha - \beta}{2\gamma}, \\
t &= \frac{|\epsilon|}{\sqrt{1 + |\epsilon|^2}}, \\
c &= \cos \theta = (1 + t^2)^{-1/2}, \\
s &= \sin \theta = ct.
\end{align*}

And,
\begin{align*}

c - s &\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} s c &\begin{pmatrix} \alpha & \gamma t \\ \beta & \gamma t \end{pmatrix} =

c - s &\begin{pmatrix} \alpha & -s \\ s & c \end{pmatrix} s c &\begin{pmatrix} \alpha - \gamma t \\ 0 \\ 0 & \beta + \gamma t \end{pmatrix}.
\end{align*}

With Jacobi rotations, two-dimensional subspaces are rotated. There are three of them:
\begin{align*}
R_{12} &= \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
R_{13} &= \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix}, \\
R_{23} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}.
\end{align*}

To use those matrices iteratively to diagonalise $A$ is the core of the Jacobi method.

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1. Input $A_0 := A$. Initialise $E_0 := \text{diag}(1,1,1)$ which will contain the eigenvectors and set the tolerances value $tol = 10^{-14}$.
2. Find the largest off diagonal element of $A_n(i,j)$,
\[ (i,j) = \arg \max |A_n(i,j)|, \ i < j. \]
3. Find $c$ and $s$ using
\[ \alpha = A(i,i), \beta = A(j,j), \gamma = A(i,j) \]
4. Rotate $A$, $A_n := R_{ij} A_n - 1 R_{ij}^T$.
5. Rotate $E$, $E_n := R_{ij}^T E_n - 1$.
6. If $\max |A_n(i,j)| < tol$ end, else repeat from step 2.

- Matrix multiplications are explicitly written out (generality vs speed)
- Quadratic convergence
- Well suited for parallelisation
- 30% faster than DIPLib (single core)
- Get code from me

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**Direction vs Orientation I**

A vector in a metric space represents a direction. In $\mathbb{R}^N$, $N - 1$ scalars are required (example). A direction points out how to get from point A to point B in $\mathbb{R}^N$. An orientation tells you to point your nose at B and have your feet down. There is a strong relationship between orientations and rotations. The natural setting for a discussion on orientations is group theory (see my thesis!) Bild: Jordglob

**Direction vs Orientation II**

The dimensionality of orientation

Of necessity, rotation matrices are ON. All eigenvalues have length 1. The minimal number of elements that are needed to describe this is $1 + 2 + \ldots + (N - 1) = N(N - 1)$ (odd dimensions)
Example I, KDE vs histogram

Example II, structure description

Example III, rotation space

Example IV, Structure Tensor

Example V, Structure Tensor

Example VI, curvature

On meshes
Summary
- Not to choose is also a choice!
- There are a few different techniques for local direction estimation.
- For larger regions, orientation can be estimated as well.
- I'd like to see more KDEs!
- There is much more to this subject!

Selected References