We observe a 2D digital image and model the image intensities in one band. Also, the resulting segmentation is generally too fuzzy (too many image pixels are classified as mixed).

Our goal is to obtain a coverage segmentation in each of the observed bands, and a matrix \( C \) provides a linear unmixing segmentation. A coverage segmentation by energy minimization fulfils
\[
\arg \min_{A \in \mathbb{R}^{N \times m}} D(A),
\]
where \( D(A) = \|I \approx AC\|_F^2 \), and \( \|X\|_F \) is the Frobenius norm (Euclidean norm) of a matrix \( X \).

Minimization of \( D(A) \) (calculus of variations) constrained to \( A \in \mathbb{R}^{N \times m} \) provides a linear unmixing segmentation.

\[
A^* = \arg \min_{A \in \mathbb{R}^{N \times m}} D(A)
\]

The lack of spatial information makes this type of coverage segmentation noise sensitive. Also, the resulting segmentation is generally too fuzzy.

Properties of coverage representations:
- homogeneous connected regions of “pure” pixels
- separated by thin layers of “mixed” pixels

Data fidelity term

\[
D(A) = \|I \approx AC\|_F^2,
\]
where \( I \) is the (expected) image value of a class \( k \) in the band \( l \). Using the introduced notation, we can, conveniently, express that \( I \) is approximately a linear mixture of the end-members as follows
\[
I \approx AC.
\]

Note: This notation suggests that the end-members \( c_{j,k} \) are position invariant. This is not necessarily the case; we allow spatially varying class representatives \( C = C(x) \). However, to not complicate notation, we write \( C \) as an \( m \times b \) matrix, and not as an \( N \times m \times b \) 3D tensor.

\[\text{1} \text{ Appropriate determination of end-members is a subject of many studies and outside the scope of this presentation.}\]
More energy terms

We add two more criteria to our (so far “too noisy” and “too fuzzy”) segmentation model:

(i) we favour a smooth boundary of each object;
(ii) we favour objects with majority of pixels classified as pure, whereas mixed pixels appear only as thin boundaries between the objects.

Criterion (i) is implemented by inclusion of the (fuzzy) perimeter of the objects as a term in the energy function to minimize. Criterion (ii) is imposed by minimizing “thickness” of boundaries over the image, and also, to some extent, minimizing overall fuzziness of the image.

These requirements are combined into the following energy function:

$$J(A) = D(A) + \mu P(A) + \nu T(A) + \xi F(A),$$

where $D, P, T, F$ are data term, overall perimeter, boundary thickness, and total image fuzziness, and $\mu, \nu, \xi \geq 0$ are weighting parameters.

Perimeter, thickness, and fuzziness

Perimeter $P(A)$ is the overall (fuzzy) perimeter of the $m$ objects of a coverage segmentation $A$:

$$P(A) = \frac{1}{2} \sum_{i=1}^{m} P(A_i).$$

Thickness We define border thickness $T$ of a coverage segmentation as:

$$T(A) = \frac{1}{2} \sum_{i=1}^{m} T(A_i),$$

where the thickness of one component $T(A_i)$ is the sum of local thickness computed for all $2 \times 2$ tiles of the image:

$$T(A_i) = \sum_{(a_1, a_2) \in \tau(A_i)} \prod_{i=1}^{4} 4\alpha_i(1 - \alpha_i).$$

Fuzziness The inclusion of an overall fuzziness term allows better control of the fuzziness in the resulting segmentation:

$$F(A) = \sum_{i=1}^{m} \sum_{j=1}^{m} 4\alpha_{ij}(1 - \alpha_{ij}).$$

Minimization

The sought coverage segmentation $A^*$ is obtained by minimizing the complete energy functional $J$ over the set of valid coverage segmentations:

$$A^* = \arg\min_{A \in \Omega_{0,m}} J(A).$$

A convex constrained large scale non-convex optimization problem.

Encouraged by good results obtained when addressing problems of similar structure and dimensionality we decided to use the Spectral Projected Gradient (SPG) method.

The SPG method requires differentiating the energy function $J(A)$. The partial derivative of $J(A)$ w.r.t. an individual coverage value $\alpha_{ij}$ is

$$\frac{\partial J(A)}{\partial \alpha_{ij}} = \frac{\partial D(A)}{\partial \alpha_{ij}} + \mu \frac{\partial P(A)}{\partial \alpha_{ij}} + \nu \frac{\partial T(A)}{\partial \alpha_{ij}} + \xi \frac{\partial F(A)}{\partial \alpha_{ij}}.$$
The method of Villa et al. (2011) performs sub-pixel classification. (SVM-based coverage segmentation is followed by spatial high resolution assignment by means of simulated annealing optimization.)

To compare our results, we generate two high resolution distributions of coverage:

1. **Stupid** method: Perform crisp classification and scale up by a factor 3
2. **Optimal** method: Distribute the coverage to best match the ground truth

This provides **lower and upper bounds** of accuracy for a possible sub-pixel assignment of the coverage values.

<table>
<thead>
<tr>
<th></th>
<th>Accuracy [%]</th>
<th>CPU time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Villa et al. 2011</td>
<td>90.05</td>
<td>58 (88 incl. SA)</td>
</tr>
<tr>
<td>Proposed</td>
<td>92.09/94.74</td>
<td>4.5</td>
</tr>
</tbody>
</table>

Further improvements to the method of Villa et al. (2011) are obtained by removing nontarget pixels from the coverage. This is done by using a target model, which is defined by excluding the nontarget classes of interest. The target model is used to calculate the coverage of each pixel, and the nontarget pixels are removed from the coverage matrix.

### Quantitative evaluation

- **Noise sensitivity**: The effect of noise on the accuracy of the coverage segmentation is assessed by adding noise to the original images. The noise is added in the form of Gaussian noise with a specified variance.

- **Evaluation**:
  - **Crisp, no noise**: The ground truth is used as the reference for evaluating the accuracy of the coverage segmentation.
  - **Alg 1, µ = 0, σ = 0.5**: The first method with specific parameters.
  - **S & L (2009b)**: The second method.

### Segmentation of hyperspectral data

- **Test on a publicly available hyperspectral data set**: The data set contains 220 bands of hyperspectral information.
- **Same data is used in Villa et al. 2011**: Allowing for a direct performance comparison.
- **Available ground truth classification is crisp**: The ground truth is used to evaluate the accuracy of the coverage segmentation.
- **Object border pixels for different noise levels**: The coverage of the object border pixels is evaluated for different noise levels.
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Further improvements … work in progress

- We observe that the perimeter term (which likes fuzzy plateaus) and the two defuzzifying terms to some extent fight each other.
- To reach a desired result, the defuzzifying terms have to be strong enough, but should not be so strong as to give a crisp output.

**A difficult balance act** which is only partly solved by the designed algorithm (with its slow increase of defuzzifying terms and a smart stopping criterion).

A new Fuzziness term

- Fuzziness should not be penalized when it appears on object boundaries.
- Scale the fuzziness term based on local “edgeness”.

\[ F(A) = \sum_{i=1}^{N} \sum_{j=1}^{N} 4\alpha_{ij}(1 - \alpha_{ij})(1 - \kappa_{ij}) , \]

where

\[ \kappa_{ij} = \max_{k \in N(i)} \min_{k \in N(j)} \alpha_{kj} \]

and \( N(i) \) is the \( 3 \times 3 \) neighbourhood of pixel \( i \).

- The new term is able to replace both previous terms \( T \) and \( F \).
- The larger \( 3 \times 3 \) neighbourhood makes processing a bit slower.
- Much improved stability w.r.t. parameter changes.
- Allows simplified algorithm and gives better results.

Quantitative evaluation
- noise sensitivity

Left: (top) Synthetic test objects. (middle) Part of object with 30% noise added. (bottom) Coverage segmentation result for 30% noise. Right: Average absolute error of coverage values of object border pixels for different noise levels. Lines show averages for 50 observations and bars indicate max and min errors.

A new Data fidelity term

- The only term in the Energy function that relates to the input image is the Data term.
- By matching an \( n \times n \) block of pixels in the segmented image \( A \) with one pixel in the input image \( I \), super resolution coverage segmentation is directly available.

Reconstruction at increased resolution
- evaluation of noise sensitivity

Left: (top) Synthetic test objects. (middle) Part of object with 30% noise added. (bottom) Coverage segmentation result for 30% noise at twice the original resolution. Right: Average absolute error of coverage values of object border pixels for different noise levels at twice the original resolution. Lines show averages for 50 observations and bars indicate max and min errors.

Segmentation at four times the original resolution, for a noise free case, 15% of added noise, and 30% of added noise, respectively.
The Coverage model

Nataša Sladoje and Joakim Lindblad

Preliminaries

Energy terms

Minimization

Evaluation

Application

Example

Further improvements

Reconstruction at increased resolution

- satellite image

(a) One band (30 out of 220) of a low resolution image obtained by averaging of $3 \times 3$ blocks in the original image

(b) Ground truth for the high resolution image, with unclassified pixels presented in black

(c) New super resolution segmentation (3 times higher resolution)

Deconvolution

- If the point spread function is larger than the size of a pixel, the linear mixture assumed in the Data term starts to be questionable.
- However, the convolution of the image data with a point spread function can be straightforwardly incorporated into the Data fidelity term.
- Using the introduced notation, this is just one more matrix multiplication (see the function `convmtx2` in Matlab).

$$D(A) = \| I - KAC \|^2, $$

Work in progress . . .

Original image with three training regions.

Promising first results with deconvolution . . .