Local minima, marginal functions, and separating hyperplanes in discrete optimization

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We prove results in optimization theory of two integer variables which correspond to fundamental results in convex analysis of real variables, viz. that a local minimum of a convex function is global; that the marginal function of a convex function is convex; and that two disjoint convex sets can be separated by a hyperplane. We show by simple examples that none of these fundamental results holds for functions which are restrictions to $\mathbf{Z}^2$ of convex functions defined on $\mathbf{R}^2$. But for a class of functions of two discrete variables called integrally convex functions there are perfect analogues of the three results.

Define a difference operator D_a for $a \in \mathbf{Z}^2$ by $D_a f(x) = f(x+a) - f(x)$, $x \in \mathbf{Z}^2$, $f\colon \mathbf{Z}^2 \to \mathbf{R}$.

A function $f\colon \mathbf{Z}^2 \to \mathbf{R}$ is said to be *integrally convex* [1, 2] if it satisfies $D_b D_a f \geq 0$ for all $(a, b) \in \mathbf{Z}^2 \times \mathbf{Z}^2$ with $a = (1, 0)$, $b = (1, -1), (1, 0), (1, 1)$ as well as $a = (0, 1)$, $b = (-1, 1), (0, 1), (1, 1)$.

If, given a point $p \in \mathbf{Z}^2$, an integrally convex function satisfies $f(x) \geq f(p)$ for all x such that $\|x - p\|_\infty \leq 1$, then it satisfies $f(x) \geq f(p)$ for all x. Actually sometimes a smaller neighborhood can suffice [2].

For any integrally convex function $f\colon \mathbf{Z}^2 \to \mathbf{R}$, its marginal function $h(x) = \inf_{y \in \mathbf{Z}} f(x, y)$, $x \in \mathbf{Z}$, is convex.

Given two integrally convex functions $f, g\colon \mathbf{Z}^2 \to \mathbf{R}$, consider the sets

$$A = \{(x, y, z) \in \mathbf{Z}^3; z \geq f(x, y)\}, \quad B = \{(x, y, z) \in \mathbf{Z}^3; -g(x, y) \geq z\}.$$

Then there exists a plane $z = H(x, y)$ separating A and B, i.e., there is an affine function $H\colon \mathbf{R}^2 \to \mathbf{R}$ such that $f \geq H|_{\mathbf{Z}^2} \geq -g$, if and only if $f_{\frac{1}{2}} + g_{\frac{1}{2}} \geq 0$, where $f_{\frac{1}{2}}\colon \mathbf{Z}^2 \cup (\mathbf{Z} + \frac{1}{2})^2 \to \mathbf{R}$ is defined for $(x, y) \in \mathbf{Z}^2$ by $f_{\frac{1}{2}}(x, y) = f(x, y)$ and

$$f_{\frac{1}{2}}(x + \tfrac{1}{2}, y + \tfrac{1}{2}) = \tfrac{1}{2} \min \left[f(x, y) + f(x+1, y+1), f(x+1, y) + f(x, y+1) \right].$$

Work on more than two variables is in progress.

References

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