Frozen history: Reconstructing the climate of the past

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Dedicated to Vladimir Maz′ya, a great mathematician and a great human being

Abstract. The ice caps on Greenland and Antarctica are huge memory banks: the temperature of the past is preserved deep in the ice. In this paper we present a mathematical model for the reconstruction of past temperatures from records of the present ones in a drilled hole.

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1. Introduction

The ice sheets on Greenland and Antarctica are huge memory banks. Dorthe Dahl-Jensen and collaborators (1998) measured the temperature in a hole in the ice cap of Greenland down to a depth of 3028.6 meters and could reconstruct the temperature on the surface of the ice during the last 50,000 years. The hole is situated at 72°35′ N, 37°38′ W and was drilled as a part of the Greenland Ice Core Project (GRIP).

The authors developed a Monte-Carlo method to fit the data and infer past climate. They made 3,300,000 forward calculations and chose the 2000 ones which gave the best fit to the temperatures recorded in the hole in 1995.

The climate of Antarctica over the longer period of 740,000 years was characterized through a new ice-core record. The core, drilled by the European Project for Ice Coring in Antarctica (EPICA 2004), came from Dome C (75°06′ S, 123°21′ E, altitude 3233 meters above sea level). The deuterium/hydrogen ratio was used as a temperature proxy, as in a more recent study of the newest ice core from Greenland, NGRIP (Steffensen et al. 2008). Thus the actual temperatures were not used in these cases. However, temperatures are routinely measured in the holes (Margareta Hansson, personal communication 2008-04-24).

The amplitude of the variations in the reconstructed temperatures from the GRIP hole varied between 23 K and 0.5 K and the precision of the measurements in the GRIP hole was 5 mK (Dahl-Jensen et al. 1998:268). Therefore some variations could be detected even at depths where the amplitude compared with the amplitude at the surface has been reduced by as much as a factor 500 or even 1000.

The purpose of the present paper is to analyze the feasibility of such a reconstruction. Since it involves solving the heat equation backwards, it is to be expected that the problem will be ill-posed. However, restricting attention to temperatures of a certain kind, a reasonably stable reconstruction is nevertheless possible.

We shall first investigate the continuity properties of a forward calculation, i.e., considering the surface temperature at all past moments as the given function and deducing the present temperature in a hole. Then we analyze the inverse operator yielding the past temperatures as a function of the present temperatures.

The main results of this paper were found in November 2002 when I was lecturing on partial differential equations at the Graduate School of Mathematics and Computing.

2. The heat equation

We consider

$$G = \{(t,x,y,z) \in \mathbf{R}^4;\ t \leqslant 0, z \leqslant \rho t\},$$

a sector in the four-dimensional space $\mathbf{R}^4$, as a model of the ice sheets on Greenland or Antarctica. Here t is the time, x and y are the horizontal coordinates, and z is the depth, counted negatively under the present ice surface.[1] The number ρ is a nonnegative constant allowing for the accumulation of snow. In fact, at the central parts of Greenland and Antarctica, the snow is kept in place, causing the surface to rise slowly. The lower strata flow out and become thinner but conserve their order. By contrast, at the periphery of the ice caps, the ice flows out into the surrounding oceans. The constant ρ is probably of the order of 10^{-9} m/s and can be taken to be zero for shorter periods.

In a more refined model, one should assume the horizontal dimensions of the ice to be bounded and restrict the depth to the range $-3028.6\,\mathrm{m} \leqslant z \leqslant 0$ in the case of Greenland. Also, one should consider the terrestrial heat flow from the underlying bedrock. However, in this first study we shall not do so; we also simplify and consider $-\infty < x, y < +\infty$ and $-\infty < z \leqslant \rho t$.

We consider *temperature functions*, by which we mean continuous complex-valued functions u on G which are of class C^2 in the interior G° of G and satisfy

[1] I apologize to geoscientists, who usually let z be positive below the surface.

the heat equation

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{2.1}$$

there. Here κ is a positive constant, called *thermal diffusivity* or *temperature conduction capacity.* For ice at $-4\,^\circ\mathrm{C}$, $\kappa = 1.04 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$. For copper this parameter is more than one hundred times larger, $1.1161 \cdot 10^{-4}\,\mathrm{m}^2/\mathrm{s}$, yielding an attenuation parameter β which is ten times smaller and the wavelength of the vertical function v ten times larger (see section 7).

In our model we assume that the temperature functions are independent of x and y, so that we actually let G_1, defined as a sector in the third quadrant,

$$G_1 = \{(t, z) \in \mathbf{R}^2; t \leqslant 0, z \leqslant \rho t\},$$

be the model domain. The heat equation reduces to

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial z^2}. \tag{2.2}$$

For each temperature function in G_1 there is a function $h(t) = u(t, \rho t)$, $t \leqslant 0$, describing the temperature on the surface of the ice ($z = \rho t$) up to the present time, and a function $v(z) = u(0, z)$, $z \leqslant 0$, describing the present temperature in a hole in the ice ($t = 0$).[2] So we have restriction mappings $u \mapsto h$ and $u \mapsto v$. We now formulate two problems.

The direct problem. *Given a function on the surface of the ice for all past moments in time, find the present temperature at all depths $z \leqslant 0$. Thus, given $h(t) = u(t, \rho t)$ for $t \leqslant 0$, find $v(z) = u(0, z)$ for $z \leqslant 0$ for a suitable temperature function u.*

The inverse problem. *Given a function in a hole at the present time, find the temperature at the surface of the ice for all moments in the past. Thus, given $v(z) = u(0, z)$ for $z \leqslant 0$, find $h(t) = u(t, \rho t)$ for $t \leqslant 0$ for a suitable temperature function u.*

The boundary of G_1 consists of the two rays

$$S_1 = \{(t, \rho t) \in \mathbf{R}^2;\ t \leqslant 0\} \text{ and } S_2 = \{(0, z) \in \mathbf{R}^2;\ z \leqslant 0\},$$

thus $\partial G_1 = S_1 \cup S_2$. This means that the two problems are concerned with the boundary values of temperature functions in G_1 and whether one can pass from the values on one ray to the other.

3. Nonuniqueness in the direct problem

It is not possible to determine uniquely the present temperature v from the surface temperature h in the past: there are many temperatures u such that $h(t) = u(t, \rho t) = 0$ for all $t \leqslant 0$.

[2] Mnemonic trick: h for horizontal or history, v for vertical.

Indeed, the function $u(t,z) = Cz$ for $\rho = 0$ and $C\left(e^{\rho(\rho t - z)/\kappa} - 1\right)$ for $\rho \neq 0$ solves the equation with $h(t) = u(t,\rho t) = 0$ for any C. So there are infinitely many solutions, but the only bounded solution of this form is $u = 0$.

More generally, we may take

$$u(t,z) = C\big(e^{\kappa\beta^2 t + \beta z} - e^{\kappa\gamma^2 t + \gamma z}\big),$$

where β is an arbitrary number and $\gamma = -\beta - \rho/\kappa$. It is bounded only when it vanishes. When $\gamma = 0$ it reduces to the first-mentioned example.

4. Uniqueness in the direct problem

Proposition 4.1. *Given a function $h \in C(\mathbf{R}_-)$, there is at most one temperature function u in G_1 such that $u(t,\rho t) = h(t)$, $t \in \mathbf{R}_-$, in the class of temperature functions satisfying*

$$(-t_1)^{-1/2} \sup_{z \leqslant \rho t_1} |u(t_1, z)| \to 0 \ \textit{as} \ t_1 \to -\infty.$$

Here $\mathbf{R}_- = \{t \in \mathbf{R};\ t \leqslant 0\}$.

So the quantity $\sup_{z\leqslant \rho t_1} |u(t_1,z)|$ may grow slowly as $t_1 \to -\infty$, but it must be finite for all t_1, meaning that the solution must be bounded on each ray $\{(t_1, z) \in \mathbf{R}^2; z \leqslant \rho t_1\}$.

Proof. We know that for any solution U defined for $T_0 < t < T_1$ and all real z, a convolution formula enables us to calculate $U(t,\cdot)$ from $U(t_1,\cdot)$ when $T_0 < t_1 < t < T_1$:

$$U(t,\cdot) = E_{t-t_1} * U(t_1,\cdot),$$

where the convolution is with respect to the space variable only. Here $E_t(z) = (4\pi t)^{-1/2}\exp(z^2/(4t))$ is the heat kernel. Hence

$$U(t,z_1) - U(t,z_2) = \int_{\mathbf{R}} \big(E_{t-t_1}(z_1 - z) - E_{t-t_1}(z_2 - z)\big) U(t_1, z) dz.$$

We estimate the difference:

$$|U(t,z_1) - U(t,z_2)| \leqslant \sup_z |U(t_1,z)| \int \big|E_{t-t_1}(z_1 - z) - E_{t-t_1}(z_2 - z)\big| dz.$$

We now fix t and let t_1 tend to $-\infty$, assuming that $T_0 = -\infty$. The integral in the last formula is equal to a quantity $K(t - t_1, z_1 - z_2)$ which tends to zero as $t - t_1$ tends to $+\infty$; more precisely, it is bounded by a constant times $(-t_1)^{-1/2}$ for $t_1 \ll 0$. This shows that $|U(t,z_1) - U(t,z_2)| = 0$ as soon as $M(t_1) = \sup_z |U(t_1,z)|$ is bounded or more generally is such that $(-t_1)^{-1/2} M(t_1) \to 0$ as $t_1 \to -\infty$. Hence U is independent of z, so we have $U(t,z) = U(t,\rho t) = h(t)$ and conclude that U is determined by h.

However, all this supposes that the solutions are defined for all real z. If a solution u is only defined for $z \leqslant \rho t$ and vanishes for $z = \rho t$, one can extend it by mirroring: we define $U(t,z) = u(t,z)$ if $z \leqslant \rho t$ and $U(t,z) = -u(t, 2\rho t - z)V(t,z)$

if $z > \rho t$. Here we choose $V(t,z) = e^{At+Bz} = e^{\rho(\rho t - z)/\kappa}$, an exponential solution satisfying $V(t,z) = 1$ when $z = \rho t$. In fact, writing H for the heat operator $\partial/\partial t - \kappa\partial^2/\partial z^2$,

$$\begin{aligned}(HU)(t,z) = &-(Hu)(t,2\rho t - z)V(t,z) - 2u_z(t,2\rho t - z)(\rho V(t,z) + \kappa V_z(t,z)) \\ &-u(t,2\rho t - z)(HV)(z,t), \qquad z > \rho t.\end{aligned}$$

Here $(Hu)(t,2\rho t - z) = (HV)(z,t) = 0$, and $\rho V + \kappa V_z$ vanishes with our choice of $A = \rho^2/\kappa$ and $B = -\rho/\kappa$. When $u(t,\rho t) = 0$, the extension is sufficiently smooth for the argument above to work; indeed the z-derivative of U from the right is equal to that of u from the left on the line $z = \rho t$. We note that $M(t_1) = \sup_{z\in\mathbf{R}} |U(t_1,z)| = \sup_{z\leqslant\rho t_1} |u(t_1,z)|$, for $|V(z,t_1)| \leqslant 1$ where $z \geqslant \rho t_1$, so the condition on the growth of u implies the same condition for the extension U.

If we have two solutions u_1 and u_2 with the same restriction $h(t) = u_1(t,\rho t) = u_2(t,\rho t)$, then we apply the above argument to $u_1 - u_2$. □

5. Nonuniqueness in the inverse problem

It is well known, thanks to A. N. Tihonov, that there is no uniqueness in the inverse problem; see, e.g., John (1991:211–213). However, two different solutions must differ by an unbounded function, indeed of very strong growth at infinity (Täcklind 1936).

6. Uniqueness in the inverse problem

We shall see that for the classes of functions we introduce, $\mathscr{V}$ and $\mathscr{H}$, we do have uniqueness in the inverse problem $v \mapsto h$.

7. Exponential solutions

An exponential function $u(t,z) = e^{At+Bz}$, where A and B are complex constants, is a solution to (2.2) if and only if $A = \kappa B^2$. The restriction of this function to the line $t = 0$ is the memory function $v(z) = u(0,z) = e^{Bz}$, and the restriction to the line $z = \rho t$ is the history $h(t) = u(t,\rho t) = e^{(A+\rho B)t}$.

Given a complex number α and real numbers β and γ we consider

$$u(t,z) = e^{(i\alpha - \rho(\beta+i\gamma))t + (\beta+i\gamma)z},$$

thus with $A = i\alpha - \rho(\beta + i\gamma)$ and $B = \beta + i\gamma$. This function satisfies

$$u(t,\rho t) = h(t) = e^{i\alpha t} \text{ and } u(0,z) = v(z) = e^{(\beta+i\gamma)z}.$$

It is a solution to the heat equation if and only if

$$\alpha = (2\kappa\beta + \rho)\gamma, \quad \gamma^2 = \beta^2 + \rho\beta/\kappa. \tag{7.1}$$

When $\rho = 0$ this simplifies to

$$\alpha = \pm 2\kappa\beta^2, \quad \gamma = \pm\beta. \tag{7.2}$$

In view of the application to temperatures, it is reasonable to assume that $h(t) = e^{i\alpha t}$ is bounded and does not tend to zero as $t \to -\infty$, which means that α should be real.

Given a nonnegative number β we therefore have two exponential solutions with damping like $e^{\beta z}$ as $z \to -\infty$ (coinciding and constant when $\beta = 0$):

$$u_1(t,z) = e^{i\alpha t+(\beta+i\gamma)z} \text{ for } \alpha \geqslant 0 \text{ and } u_2(t,z) = e^{i\alpha t+(\beta-i\gamma)z} \text{ for } \alpha \leqslant 0, \tag{7.3}$$

where α is given by (7.1). Their real and imaginary parts are, respectively

$$\begin{aligned}\operatorname{Re} u_1(t,z) &= e^{\beta z}\cos(\alpha t+\gamma z), \quad \operatorname{Im} u_1(t,z) = e^{\beta z}\sin(\alpha t+\gamma z),\\ \operatorname{Re} u_2(t,z) &= e^{\beta z}\cos(\alpha t-\gamma z), \quad \operatorname{Im} u_2(t,z) = e^{\beta z}\sin(\alpha t-\gamma z).\end{aligned}$$

These functions have a *temporal period* $p = 2\pi/|\alpha|$, an *attenuation parameter* equal to $\beta \geqslant 0$ describing how the temperature tapers off as we go downwards, and a *spatial period* for the argument equal to $q = 2\pi/|\gamma|$.

When $\rho = 0$, (7.3) simplifies to

$$u_1(t,z) = e^{i\alpha t+\beta(1+i)z} \text{ for } \alpha \geqslant 0 \text{ and } u_2(t,z) = e^{i\alpha t+\beta(1-i)z} \text{ for } \alpha \leqslant 0, \tag{7.4}$$

where α is given by (7.2).

Conversely, given a real number α, there are unique numbers $\beta \geqslant 0$ and $|\gamma|$ such that (7.1) is satisfied. If $\rho = 0$, this simplifies to

$$\begin{aligned}\beta &= \sqrt{\frac{\alpha}{2\kappa}} \geqslant 0, \quad \gamma = \beta \geqslant 0, \text{ when } \alpha \geqslant 0, \text{ and}\\ \beta &= \sqrt{\frac{-\alpha}{2\kappa}} > 0, \quad \gamma = -\beta < 0, \text{ when } \alpha < 0.\end{aligned} \tag{7.5}$$

(The negative square roots will be disregarded.) In general β is the solution of an equation of degree four.

The quotients

$$\frac{|\alpha|}{2\kappa\beta^2} = \sqrt{1+\frac{\rho}{\kappa\beta}}\left(1+\frac{\rho}{2\kappa\beta}\right) \quad \text{and} \quad \frac{\gamma^2}{\beta^2} = 1+\frac{\rho}{\kappa\beta}$$

both increase with ρ when β is fixed, but the quotient between them,

$$\frac{q^2}{4\pi\kappa p} = \frac{|\alpha|}{2\kappa\gamma^2} = \frac{|\alpha|}{2\kappa\beta^2}\Big/\frac{\gamma^2}{\beta^2} = \frac{1}{|\gamma|}\left(\beta+\frac{\rho}{2\kappa}\right) = \frac{1+\frac{\rho}{2\kappa\beta}}{\sqrt{1+\frac{\rho}{\kappa\beta}}} = \sqrt{1+\frac{\rho^2}{4\kappa\beta(\kappa\beta+\rho)}}$$

does not vary so much with ρ for fixed β. Because of this the relation between the temporal period $p = 2\pi/|\alpha|$ and the spatial period $q = 2\pi/|\gamma|$ is not so sensitive for different values of ρ; on the other hand, given a temporal period p, the attenuation parameter β becomes smaller when ρ increases.

As an example we list the spatial periods for different temporal periods. We take $\rho = 0$, which in particular means that the attenuation parameter β is equal to $\gamma = 2\pi/q$. The numerical values in the table below will be different if $\rho > 0$, but the table will nevertheless give an idea of what we can expect. The attenuation over one half spatial period is $e^{-\pi} \approx 0.0432$ (at this depth, the variation is opposite to that at the surface); over one spatial period it is $e^{-2\pi} \approx 0.001867$; over one and a half spatial period $e^{-3\pi} \approx 0.0000807$. Thus the amplitude at the depth of one spatial period (where the variation is in phase with that at the surface) is just barely measurable.

At the depth of 1.06 meters the diurnal variation is reduced by a factor of $e^{2\pi} \approx 535$, but the amplitude of the annual variation is still quite large,

$$\exp(-2\pi/\sqrt{365.2422}) \approx 0.72.$$

Thus the variation due to a longer period in h is measurable at a greater depth even though, at the surface, the amplitudes of variations of shorter periods may be much larger. This is why the ice can serve as memory.

At the depths of

$$d_{0.01} = \frac{\log 100}{2\pi} q \approx 0.733q \text{ and } d_{0.001} = \frac{\log 1000}{2\pi} q \approx 1.10q$$

the amplitude of an oscillation is 1/100 and 1/1000, respectively, of the amplitude at the surface.

In the table below, we list various periods p of a wave $h(t) = e^{i\alpha t}$; its frequency $\alpha = 2\pi/p$ is measured in Hertz, $\mathrm{Hz} = \mathrm{s}^{-1}$, and the attenuation parameter $\beta = \sqrt{\alpha/(2\kappa)} \approx 1738/\sqrt{p}$ is expressed in inverse meters, m^{-1}. The thermal diffusivity κ is taken as $1.04 \cdot 10^{-6}\mathrm{m}^2/\mathrm{s}$. The last column gives the spatial period q of the argument, $q = 2\pi/\beta = \sqrt{4\pi\kappa p} \approx 0.003615\sqrt{p}$.

Temporal period p	$\alpha = 2\pi/p$	$\beta = \sqrt{\alpha/(2\kappa)}$	Spatial period $q = \sqrt{4\pi\kappa p}$
s	$\mathrm{Hz} = \mathrm{s}^{-1}$	m^{-1}	m
$24\,\mathrm{h} = 8.64 \cdot 10^4$	$7.27 \cdot 10^{-5}$	5.91	1.06
365.2422 days $= 3.1557 \cdot 10^7$	$1.99 \cdot 10^{-7}$	0.309	20.3
10 years $= 3.1557 \cdot 10^8$	$1.99 \cdot 10^{-8}$	$9.78 \cdot 10^{-2}$	64
100 years $= 3.1557 \cdot 10^9$	$1.99 \cdot 10^{-9}$	$3.09 \cdot 10^{-2}$	203
1000 years $= 3.1557 \cdot 10^{10}$	$1.99 \cdot 10^{-10}$	$9.78 \cdot 10^{-3}$	642
10,000 years $= 3.1557 \cdot 10^{11}$	$1.99 \cdot 10^{-11}$	$3.09 \cdot 10^{-3}$	2033
100,000 years $= 3.1557 \cdot 10^{12}$	$1.99 \cdot 10^{-12}$	$9.78 \cdot 10^{-4}$	6425

For instance, let us consider the diurnal variation of the surface temperature, thus with $\alpha = 2\pi/86{,}400\,\mathrm{s}^{-1}$. This gives $\beta \approx 5.91\,\mathrm{m}^{-1}$, which means that the amplitude at the surface is reduced by a factor of 535 at the depth of 1.06 meters.

For the annual variation we have

$$\alpha \approx 2\pi/(3.1557 \cdot 10^7)\,\mathrm{s}^{-1} \text{ and } \beta \approx 0.309\,\mathrm{m}^{-1},$$

meaning that the amplitude is reduced by a factor of 535 at the depth of 20.3 meters.

For the period of 80,000 years, which is perhaps typical for the ice ages, we get $\alpha = 2\pi/(3.15576 \cdot 10^7 \cdot 8 \cdot 10^4) \approx 2.49 \cdot 10^{-12}$ and $\beta \approx 0.001094\,\mathrm{m}^{-1}$, meaning that the amplitude is reduced by a factor of 23 at the depth of 2872 meters.

The fact that the spatial period is proportional to the square root of the temporal period is of course crucial, given the time period we want to study and the dimensions of the ice cap.

The amplitudes $b_{\beta+i\gamma}$ of the recorded temperatures (see (8.4)) varied in the interval $[0.25, 25]$ measured in kelvin, thus at most by a factor of 100, and the spatial frequences β vary in the intervals indicated in the table. However, the diurnal and annual variations are negligible at depths larger than 20 meters, so the interesting periods are those from say 20 years to 50,000 years, yielding values for β in the interval $[0.0014, 0.07]$ measured in inverse meters, and spatial periods in the interval $[90, 4488]$ measured in meters.

8. Generalized trigonometric polynomials

We may combine the solutions found in section 7 as follows. The function

$$u(t,z) = \sum a_\alpha e^{(i\alpha-\rho(\beta+i\gamma))t+(\beta+i\gamma)z}, \qquad (t,z) \in \mathbf{R}^2, \tag{8.1}$$

where α, β and γ are related as in (7.1) and only finitely many of the coefficients a_α are nonzero, is a bounded solution of the heat equation, and the history and memory functions become

$$h(t) = \sum_{\alpha\in\mathbf{R}} a_\alpha e^{i\alpha t}, \quad t \leqslant 0; \qquad v(z) = \sum a_\alpha e^{(\beta+i\gamma)z}, \quad z \leqslant 0. \tag{8.2}$$

The function h is real valued if and only if $a_{-\alpha} = \overline{a_\alpha}$; in that case, $h(t) = a_0 + 2\sum_{\alpha>0} \operatorname{Re}(a_\alpha e^{i\alpha t}) = a_0 + 2\sum_{\alpha>0} \big(\operatorname{Re} a_\alpha \cos(\alpha t) - \operatorname{Im} a_\alpha \sin(\alpha t)\big)$.

The variation of the temperature over a couple of years may be described using just two periods, 24 hours and one year, or perhaps a little better with four periods, 12 hours, 24 hours, half a year and one year. To describe climate changes we need some longer periods, say from 20 years to 80,000 years.

Definition 8.1. Let $\mathscr{U}(G_1)$ or just $\mathscr{U}$ denote the space of all generalized trigonometric polynomials restricted to G_1,

$$u(t,z) = \sum_{\alpha\in\mathbf{R}} a_\alpha e^{i\alpha t+(\beta+i\gamma)z} = \sum_{\beta,\gamma\in\mathbf{R}} b_{\beta+i\gamma} e^{i\alpha t+(\beta+i\gamma)z}, \qquad (t,z) \in G_1,$$

where α, β, γ are related by (7.1) and the coefficients are related by $a_\alpha = b_{\beta+i\gamma}$ for these triples (α, β, γ). In the case when $\rho = 0$ we can have $b_{\beta+i\gamma} \neq 0$ only if $\gamma = \pm\beta$.

Here each attenuation parameter $\beta = \beta_\alpha$ is determined from α by (7.1), and all but finitely many of the coefficients $a_\alpha = b_{\beta+i\gamma}$ are zero. Given a function

$w \colon \mathbf{R} \to]0, +\infty[$, called the weight, we define the $\mathscr{U}$-norm of u as

$$\|u\|_{\mathscr{U}} = \sum_{\alpha} w(\alpha)|a_\alpha| = \sum_{\beta,\gamma} \tilde{w}(\beta + i\gamma)|b_{\beta+i\gamma}|.$$

Here

$$\tilde{w}(\beta + i\gamma) = w(\alpha) \text{ when } \alpha, \beta, \gamma \text{ are related by (7.1).} \tag{8.3}$$

It is sometimes convenient to assume, given the frequencies, that $\sum_\alpha w(\alpha) = \sum_{\beta,\gamma} \tilde{w}(\beta + i\gamma) = 1$, where the sum is over all real numbers α such that a_α is nonzero.

Definition 8.2. Let $\mathscr{H}(\mathbf{R}_-)$ or $\mathscr{H}$ denote the space of all trigonometric polynomials restricted to $\mathbf{R}_-$,

$$h(t) = \sum_{\alpha \in \mathbf{R}} a_\alpha e^{i\alpha t}, \qquad t \in \mathbf{R}_-,$$

where the α are real numbers and all but finitely many of the coefficients a_α are zero. We define the $\mathscr{H}$-norm of h as

$$\|h\|_{\mathscr{H}} = \sum_{\alpha \in \mathbf{R}} w(\alpha)|a_\alpha|.$$

Definition 8.3. Let $\mathscr{V}(\mathbf{R}_-)$ or $\mathscr{V}$ denote the space of all restrictions of generalized trigonometric polynomials to $\mathbf{R}_-$,

$$v(z) = \sum_{\beta,\gamma \in \mathbf{R}} b_{\beta+i\gamma} e^{(\beta+i\gamma)z}, \qquad z \in \mathbf{R}_-, \tag{8.4}$$

where all but finitely many of the coefficients $b_{\beta+i\gamma}$ are zero. We define the $\mathscr{V}$-norm of v as

$$\|v\|_{\mathscr{V}} = \sum_{\beta,\gamma} \tilde{w}(\beta + i\gamma)|b_{\beta+i\gamma}|,$$

where the weight $\tilde{w}$ is related to w by (8.3).

For the definitions of the norms to make sense, it is necessary to show that the coefficients are uniquely defined by the function values.

9. Finding the coefficients of a past temperature function

Proposition 9.1. *The coefficients of a function $h \in \mathscr{H}$ are given by the formula*

$$a_\alpha = \lim_{|I| \to +\infty} \frac{1}{|I|} \int_I h(t) e^{-i\alpha t} dt, \qquad \alpha \in \mathbf{R}. \tag{9.1}$$

Here $I = [r, s]$ is a subinterval of $\mathbf{R}_-$; its length $s - r$ is denoted by $|I|$ and assumed to be positive. This shows that $|a_\alpha| \leqslant \|h\|_\infty$.

Proof. We can find the coefficients a_θ, $\theta \in \mathbf{R}$, of h using the formula

$$\frac{1}{|I|}\int_I h(t)e^{-i\theta t}dt = a_\theta + \sum_{\alpha\neq\theta}\frac{a_\alpha}{|I|}\int_I e^{i(\alpha-\theta)t}dt, \qquad \theta \in \mathbf{R}.$$

Each of the terms after the first one tends to zero as $|I| \to +\infty$. □

10. Finding the coefficients of a memory function

Proposition 10.1. *For simplicity we consider now only the case $\rho = 0$. To find the coefficients $b_{\beta+i\gamma} = b_{\beta\pm i\beta}$ of a function $v \in \mathscr{V}$ we first extend it to the whole complex plane as an entire function w, thus $w(z)$ is given by the same formula for all $z \in \mathbf{C}$, and $w(z) = v(z)$ when $z \leqslant 0$. We then define*

$$v_1(z) = w\big(\tfrac12(1+i)z\big) = \sum_{\beta\geqslant 0} b_{\beta+i\beta}e^{i\beta z} + \sum_{\beta>0} b_{\beta-i\beta}e^{\beta z}, \qquad z \in \mathbf{R}_-, \text{ and}$$

$$v_2(z) = w\big(\tfrac12(1-i)z\big) = \sum_{\beta\geqslant 0} b_{\beta+i\beta}e^{\beta z} + \sum_{\beta>0} b_{\beta-i\beta}e^{-i\beta z}, \qquad z \in \mathbf{R}_-.$$

The coefficients are given by the formulas

$$b_{\theta+i\theta} = \lim_{|I|\to+\infty}\frac{1}{|I|}\int_I v_1(z)e^{-i\theta z}dz, \qquad \theta \geqslant 0, \text{ and}$$

$$b_{\theta-i\theta} = \lim_{|I|\to+\infty}\frac{1}{|I|}\int_I v_2(z)e^{i\theta z}dz, \qquad \theta > 0.$$

Here $I = [r, s]$ is an interval with $r < s \leqslant 0$.

Proof. We obtain

$$\frac{1}{|I|}\int_I v_1(z)e^{-i\theta z}dz$$
$$= b_{\theta+i\theta} + \sum_{\substack{\beta\geqslant 0\\ \beta\neq\theta}}\frac{1}{|I|}\int_I b_{\beta+i\beta}e^{i(\beta-\theta)z}dz + \sum_{\beta>0}\frac{1}{|I|}\int_I b_{\beta-i\beta}e^{(\beta-i\theta)z}dz.$$

It is clear that all terms except the first tend to zero as $|I| \to +\infty$. Finding the coefficients $b_{\theta-i\theta}$ is similar. □

Another way to find the coefficients of a function $v \in \mathscr{V}$ is this:

Proposition 10.2. *First we note that $b_0 = \lim_{z\to-\infty} v(z)$. Assume that the smallest positive value of β for which $b_{\beta+i\beta}$ or $b_{\beta-i\beta}$ is nonzero is β_1. Then*

$$b_{\beta_1+i\beta_1} = \lim_{z\to-\infty}(v(z)-b_0)e^{-\beta_1(1+i)z} \text{ and } b_{\beta_1-i\beta_1} = \lim_{z\to-\infty}(v(z)-b_0)e^{-\beta_1(1-i)z}.$$

Let β_2 be the smallest frequency after β_1. Then

$$b_{\beta_2+i\beta_2} = \lim_{z\to-\infty}\Big(v(z) - b_0 - b_{\beta_1+i\beta_1}e^{\beta_1(1+i)z} - b_{\beta_1-i\beta_1}e^{\beta_1(1-i)z}\Big)\, e^{-\beta_2(1+i)z},$$

$$b_{\beta_2-i\beta_2} = \lim_{z\to-\infty}\Big(v(z) - b_0 - b_{\beta_1+i\beta_1}e^{\beta_1(1+i)z} - b_{\beta_1-i\beta_1}e^{\beta_1(1-i)z}\Big)\, e^{-\beta_2(1-i)z}.$$

By repeated use of these formulas, all coefficients can be determined.

While the two last lemmas give the coefficients in the expansion of v theoretically, they are not suited for calculations. It is desirable to find a formula which gives at least approximately the values of the coefficients from information contained in an interval of finite length.

11. An isometry

Theorem 11.1. *The restriction mappings*

$$\mathscr{U} \ni u \mapsto u|_{\{(t,\rho t);t\in\mathbf{R}_-\}} = h \in \mathscr{H} \text{ and } \mathscr{U} \ni u \mapsto u|_{\{0\}\times\mathbf{R}_-} = v \in \mathscr{V}$$

are isometries.

Hence the mappings $\mathscr{H} \ni h \mapsto v \in \mathscr{V}$ and $\mathscr{V} \ni v \mapsto h \in \mathscr{H}$ are also isometries.

We thus have $\|u\|_{\mathscr{U}} = \|h\|_{\mathscr{H}} = \|v\|_{\mathscr{V}}$ if h and v are the restrictions of u. Therefore we also have isometries $h \mapsto v$ and $v \mapsto h$.

The problem is now moved over to a study of the norm $\|v\|_{\mathscr{V}}$ and a comparison between it and other norms. This we shall do in the next section.

We shall see in Theorems 14.1 and 15.1 that, if we use L^∞-norms, the problem $h \mapsto v$ is well-posed under the topologies used and for generalized trigonometric polynomials h and v as long as the number of terms is bounded and the frequencies are well separated.

We may also estimate the L^1-norm of v in terms of the L^∞ norm of h. The constant term must be treated separately.

Theorem 11.2. *Let h and v be given as in Definitions 8.2, 8.3 and be related as in* (8.2)*. Then*

$$\|v - a_0\|_1 \leqslant C\|h - a_0\|_\infty, \tag{11.1}$$

where

$$C = \sqrt{2\kappa}\sum_{\alpha\neq 0}|\alpha|^{-1/2};$$

the sum being extended over all the finitely many $\alpha \neq 0$ such that $a_\alpha \neq 0$.

Proof. We have

$$\|v - a_0\|_1 \leqslant \sum_{\beta\neq 0}(|b_{\beta+i\beta}| + |b_{\beta-i\beta}|)\int_{-\infty}^0 e^{\beta z}dz = \sum_{\beta\neq 0}(|b_{\beta+i\beta|} + |b_{\beta-i\beta}|)\beta^{-1}.$$

In view of the fact that $|b_{\beta+i\beta}| + |b_{\beta-i\beta}| = |a_\alpha|$, that $\beta^{-1} = \sqrt{2\kappa/|\alpha|}$, and that $|a_\alpha| \leqslant \|h - a_0\|_\infty$ for $\alpha \neq 0$, the last expressing is not larger than

$$\sum_{\alpha\neq 0} |a_\alpha|\beta^{-1} \leqslant \|h - a_0\|_\infty \sqrt{2\kappa} \sum_{\alpha\neq 0} |\alpha|^{-1/2},$$

where the sum is extended over the finitely many real numbers α such that $\alpha \neq 0$ and $a_\alpha \neq 0$. □

12. The direct problem

Theorem 12.1. *Given a temperature history $h \in \mathscr{H}$ of the temperature $h(t)$, $t \leqslant 0$, at the surface for all past moments in time, there is a unique memory $v = \mathbf{M}(h) \in \mathscr{V}$ of that history giving the temperature at the present time at all depths in the hole.*

The memory $\mathbf{M}(h)$ is obtained in the following way. First the set of coefficients $(a_\alpha)_\alpha$ of h is determined by formula (9.1), and then the coefficients $(b_{\beta+i\gamma})_{\beta,\gamma}$ using the relations (7.1). Finally v is synthesized as (8.4). The mapping $\mathbf{M}\colon \mathscr{H} \to \mathscr{V}$ is an isometry. We can then combine it with the injection $J\colon \mathscr{V} \to L^\infty(\mathbf{R}_-)$, which is of norm one. The mapping $J \circ \mathbf{M}\colon \mathscr{H} \to L^\infty(\mathbf{R}_-)$ is linear, injective, and continuous.

We shall see that the norm $\|\cdot\|_{\mathscr{H}}$ used in $\mathscr{H}$ is equivalent to the L^∞-norm over a bounded interval under the hypotheses stated in Theorem 14.1.

13. The inverse problem

Theorem 13.1. *Given data $v \in \mathscr{V}$ of the temperature $v(z)$, $z \leqslant 0$, at all depths in the hole at the present time, there is a unique past $h = \mathbf{P}(v) \in \mathscr{H}$ giving the temperature $h(t)$ at the surface of the ice at all past moments in time.*

The history $\mathbf{P}(v)$ is obtained in the following way. First the two functions v_1 and v_2 are determined using analytic continuation. Then the indexed families of coefficients $(b_{\beta+i\beta})_\beta$ and $(b_{\beta-i\beta})_\beta$ are determined, as well as the family of coefficients $(a_\alpha)_\alpha$. Finally h is synthesized as in (8.2). The mapping $\mathbf{P}\colon \mathscr{V} \to \mathscr{H}$ is the inverse of $\mathbf{M}$ and thus also an isometry.

There are three difficulties in the model constructed so far.

The first difficulty in the determination of $\mathbf{P}(v)$ lies in the analytic continuation giving v_1, v_2 from v. All other steps in the construction are well defined mappings which have reasonable continuity properties. Analytic continuation, on the other hand, is a notoriously ill-posed problem. We have concentrated all the difficulties of the inverse problem into the mappings $v \mapsto v_1, v_2$.

Second, the norms defined require the functions to be known over the unbounded interval $\mathbf{R}_-$. It is desirable to replace them by norms using only values in a bounded subinterval I.

Third, in applications it is necessary to find a good approximation $v \in \mathscr{V}$ to given temperature measurements $V(z_1), V(z_2), \ldots, V(z_P)$ at finitely many points z_p, $p = 1, 2, \ldots, P$.

Concerning the inverse problem $v \mapsto h$, that of reconstructing the temperature in the past from knowledge of the temperature in the hole at present, we may ask whether it is possible to turn (11.1) around and obtain an inequality like

$$\|h\|_\infty \leqslant C\|v\|_1 \tag{13.1}$$

could hold. But this is not so: take $h(t) = e^{i\alpha t}$, $v(z) = e^{\beta(1+i)z}$. Then $\|h\|_\infty = 1$ while $\|v\|_1 = 1/\beta \to 0$ as $\alpha \to +\infty$. This shows that (13.1) does not hold. However, this leaves open the possibility that (13.1) could hold if we assume that the frequencies α which build up h are bounded.

14. Other norms for the space of past temperature functions

We have seen that there is an isometry $h \mapsto v$ between the past temperature functions $h \in \mathscr{H}$ and the memory functions $v \in \mathscr{V}$. While the norm in $\mathscr{H}$ is rather natural, we have used in the space $\mathscr{V}$ a rather elusive norm. The purpose of the present section and the next one is to compare these norms with more easily accessible ones.

Example. We note that

$$h_{\alpha,\varepsilon}(t) = \frac{1}{\varepsilon}e^{i(\alpha+\varepsilon)t} - \frac{1}{\varepsilon}e^{i\alpha t}, \qquad t \in \mathbf{R}_-,\ \alpha \in \mathbf{R},\ \varepsilon \neq 0,$$

is a trigonometric polynomial of norm $\|h_{\alpha,\varepsilon}\|_{\mathscr{H}} = \|h_{\alpha,\varepsilon}\|_\infty = 2/\varepsilon$, $\varepsilon \neq 0$. However, for every bounded subinterval $I = [r, s]$ of $\mathbf{R}_-$ with $r < s \leqslant 0$, the norm $\|h_{\alpha,\varepsilon}|_I\|_\infty$ tends to a finite limit as ε tends to zero, viz. $|r|$. Indeed, $h_{\alpha,\varepsilon} \to ite^{i\alpha t}$, an exponential polynomial, as $\varepsilon \to 0$, uniformly when t is bounded. □

The example shows that there is no estimate $\|h\|_{\mathscr{H}} \leqslant C\|h|_I\|_\infty$ for any bounded interval I, and also that the class $\mathscr{H}$ is not closed under uniform convergence on bounded sets. To make it a closed subspace of $C(\mathbf{R}_-)$ we would have to consider exponential polynomials, i.e., exponential functions with polynomials as coefficients, just as in the theory of ordinary differential equations. However, if we keep the frequencies apart, the situation is different:

Theorem 14.1. *For all functions $h \in \mathscr{H}$, $h(t) = \sum a_\alpha e^{i\alpha t}$, $t \in \mathbf{R}_-$, we have an estimate*

$$\|h\|_\infty \leqslant A\|h\|_{\mathscr{H}}, \quad \textit{where} \quad A = \Big(\inf_\alpha w(\alpha)\Big)^{-1}.$$

Here the infimum is taken over the finitely many α such that $a_\alpha \neq 0$.

Conversely, if the interval $I \subset \mathbf{R}_-$ is long enough and the frequencies are kept apart, then

$$\|h\|_{\mathscr{H}} \leqslant C\|h|_I\|_\infty$$

for some constant C; more precisely

$$(1-c)\|h\|_{\mathscr{H}} \leqslant \sum_\theta w(\theta)\|h|_I\|_\infty, \text{ where } c = \frac{1}{|I|}\sum_\theta w(\theta)\sup_{\alpha\neq\theta}\frac{2}{w(\alpha)|\alpha-\theta|}.$$

If $|I|$ is so large that $c<1$, we obtain an estimate. Here the supremum and the sum are taken only over those frequencies α and θ such that the coefficients a_α, a_θ do not vanish.

If in particular $w(\alpha)=1$ for all α, then it is enough that $|I|$ be larger than $2n/\gamma$, where

$$\gamma = \inf_{\alpha,\theta,\alpha\neq\theta}|\alpha-\theta| > 0,$$

and where n is the number of frequencies α such that $a_\alpha \neq 0$.

For applications it is of course important that we have the supremum over a bounded interval I rather than all of $\mathbf{R}_-$ in this estimate. So in the set $\mathscr{H}_{n,\gamma}$ of all functions in $\mathscr{H}$ with at most n frequencies and $|\alpha-\theta| \geqslant \gamma > 0$ when $\alpha \neq \theta$, the two norms are equivalent. However, $\mathscr{H}_{n,\gamma}$ is not a vector space.

For the proof of the theorem we shall need the following lemma.

Lemma 14.2. *Let us define an entire function φ by $\varphi(\zeta) = \int_{-1}^0 e^{\zeta t}dt$; equivalently*

$$\varphi(\zeta) = \begin{cases} \dfrac{1-e^{-\zeta}}{\zeta}, & \zeta \in \mathbf{C}\smallsetminus\{0\}; \\ 1, & \zeta = 0. \end{cases}$$

Then

$$|\varphi(\zeta)| \leqslant \frac{1}{\operatorname{Re}\zeta}, \qquad \zeta\in\mathbf{C}, \operatorname{Re}\zeta > 0, \text{ and}$$

$$|\varphi(\zeta)| \leqslant \frac{1+e^{-\operatorname{Re}\zeta}}{|\zeta|}, \qquad \zeta \in \mathbf{C}\smallsetminus\{0\}.$$

In particular $|\varphi(i\eta)| \leqslant 2/|\eta|$, and, since $|\varphi(i\eta)| \leqslant 1$,

$$|\varphi(i\eta)| \leqslant \min\left(1, \frac{2}{|\eta|}\right)$$

with equality when $\eta = 0$ or $\eta \in \pi(2\mathbf{Z}+1)$.

The function φ appears in a natural way when we calculate the mean values of exponential functions. For any interval $I=[r,s]$ of length $|I| = s-r>0$ and all complex numbers ζ we have

$$\frac{1}{|I|}\int_I e^{\zeta t}dt = \frac{1}{s-r}\int_r^s e^{\zeta t}dt = e^{\zeta s}\varphi(|I|\zeta).$$

The estimates in the lemma are easily proved.

Proof of Theorem 14.1. The first part of the theorem is easy to prove: we have

$$|h(t)| \leqslant \sum_\alpha |a_\alpha| = \sum_\alpha w(\alpha)|a_\alpha|w(\alpha)^{-1} \leqslant \|h\|_{\mathscr{H}}\sup_\alpha w(\alpha)^{-1}.$$

For the second part we take the mean value of $h(t)e^{-\theta t}$ over an interval $I = [r, s] \subset \mathbf{R}_-$ to be determined later. This gives

$$M_{I,\theta} = \frac{1}{|I|}\int_I h(t)e^{-i\theta t}dt = a_\theta + \sum_{\alpha\neq\theta} a_\alpha e^{i(\alpha-\theta)s}\frac{1-e^{-i|I|(\alpha-\theta)}}{i|I|(\alpha-\theta)}$$
$$= a_\theta + \sum_{\alpha\neq\theta} a_\alpha e^{i(\alpha-\theta)s}\varphi(i|I|(\alpha-\theta)),$$

where $\varphi\colon \mathbf{C}\to\mathbf{C}$ is the function in the lemma.

We may now estimate

$$|M_{I,\theta} - a_\theta| \leqslant \sum_{\alpha\neq\theta} |a_\alpha\varphi(i|I|(\alpha-\theta))| = \sum_{\alpha\neq\theta} w(\alpha)|a_\alpha|\frac{|\varphi(i|I|(\alpha-\theta))|}{w(\alpha)} \leqslant \frac{c_\theta}{|I|}\|h\|_{\mathscr{H}}$$

if we define

$$c_\theta = |I|\sup_{\alpha\neq\theta}\frac{|\varphi(i|I|(\alpha-\theta))|}{w(\alpha)} \leqslant \sup_{\alpha\neq\theta}\frac{2}{w(\alpha)|\alpha-\theta|},$$

so that

$$|a_\theta| \leqslant |M_{I,\theta}| + |M_{I,\theta} - a_\theta| \leqslant |M_{I,\theta}| + c_\theta|I|^{-1}\|h\|_{\mathscr{H}} \leqslant \|h|_I\|_\infty + c_\theta|I|^{-1}\|h\|_{\mathscr{H}},$$

leading to

$$w(\theta)|a_\theta| \leqslant w(\theta)\|h|_I\|_\infty + c_\theta|I|^{-1}w(\theta)\|h\|_{\mathscr{H}}.$$

Summing over all θ we obtain

$$\|h\|_{\mathscr{H}}\left(1 - \frac{1}{|I|}\sum_\theta c_\theta w(\theta)\right) \leqslant \left(\sum_\theta w(\theta)\right)\|h|_I\|_\infty.$$

□

Example. Let us take four frequencies, say with temporal periods $p_j = 2000$, 4000, 8000, 16,000 years, corresponding to frequencies $\alpha_j = 2\pi/p_j = 2\pi/2000$, $2\pi/4000, 2\pi/8000, 2\pi/16{,}000$ years^{-1}. Then

$$\frac{1}{\gamma} = \sup_{j\neq k}\frac{1}{|\alpha_j - \alpha_k|} = \frac{1}{2\pi}\sup_{j\neq k}\frac{p_j p_k}{p_j - p_k} = \frac{16{,}000}{2\pi}\text{ years},$$

so that an interval of length $|I| > 2n/\gamma \approx 20{,}372$ years suffices to define an equivalent norm.

15. Other norms for the space of memory functions

We would like to have a result like Theorem 14.1 also for the memory functions, thus to estimate the $\mathscr{V}$-norm by the supremum norm over a bounded interval. This problem is subtler because of the presence of the damping factors $e^{\beta z}$. As an example we present the following result, which however needs a strong separation of the attenuation parameters. There is probably a trade-off between the length of the interval needed in an estimate and the separation of the attenuation parameters which has to be assumed; the following result is just an instance of this.

Theorem 15.1. *For any function $v \in \mathscr{V}$ of the form*

$$v(z) = b_0 + \sum_1^n \left(b_j e^{(\beta_j + i\beta_j)z} + c_j e^{(\beta_j - i\beta_j)z} \right), \qquad z \leqslant 0,$$

with attenuation parameters $\beta_0 = 0 < \beta_1 < \cdots < \beta_n$ and coefficients $b_0, \dots, b_n$, $c_1, \dots, c_n$, we have an estimate

$$\|v\|_\infty \leqslant \|v\|_{\mathscr{V}}$$

if we use the weights $w_j = 1$ for all j. Conversely, assume that the function is real valued, so that $b_0 \in \mathbf{R}$ and $c_j = \overline{b}_j$, and that the parameters are well separated in the sense that $\beta_j \leqslant \sigma\beta_{j+1}$ for some number $\sigma < \frac{1}{3}$, or, a little weaker, $\sigma < \frac{1}{3}(1 + 2\sigma^n)$. Then

$$\|v - b_0\|_{\mathscr{V}} \leqslant \frac{1}{\cos\theta} \|v|_I - b_0\|_\infty,$$

where $\theta = \sigma\pi(1 - \sigma^{n-1})/(1 - \sigma) < \pi/2$ and I is the interval $I = [s_n, 0]$ with

$$s_n = -\frac{2\pi}{\beta_1}\frac{1 - \sigma^n}{1 - \sigma} \geqslant -\frac{2\pi}{\beta_1}\frac{1}{1 - \sigma} > -\frac{3\pi}{\beta_1},$$

and the weights are defined by

$$w_j = e^{\beta_j s_n}.$$

From this weight we can pass to other weights.

Proof. The first statement is easy to prove. For the second, we may assume that $b_0 = 0$.

We note that, given any $\theta \in \,]0, \pi/2[$, the inequality $\operatorname{Re}\zeta \geqslant (\cos\theta)|\zeta|$ holds for complex numbers ζ when $\zeta/|\zeta|$ lies on an arc on the unit circle of length 2θ, occupying a fraction $\tau = \theta/\pi < \frac{1}{2}$ of the whole circumference. Using this idea, we see that a term

$$b_j e^{(\beta_j + i\beta_j)z} + c_j e^{(\beta_j - i\beta_j)z} = 2e^{\beta_j z} \operatorname{Re}\left(b_j e^{i\beta_j z} \right)$$

is at least equal to $2\cos\theta\, e^{\beta_j z}|b_j|$ on a union of intervals $M_j = \bigcup_{k \in \mathbf{Z}} M_j(k)$, each $M_j(k)$ being of length $2\pi\tau/\beta_j = \tau q_j$ and appearing periodically with a period of $q_j = 2\pi/\beta_j$, thus $M_j(k) = M_j(0) + kq_j$, $k \in \mathbf{Z}$. Here $q_j = 2\pi/\beta_j$ are the the spatial periods, satisfying $q_1 > q_2 > \cdots > q_n$ and $\sigma q_{j-1} \geqslant q_j$.

The idea is to find a point $s_n \leqslant 0$ in the intersection $\bigcap_j M_j$. Then the interval $I = I_n = [s_n, 0]$, in fact even $[s_n, s_n]$, can serve in our conclusion.

Define $s_0 = 0$ and let s_1 be the right endpoint of the interval $M_1(k_1)$ which is contained in $\mathbf{R}_-$ but such that $M_1(k_1 + 1)$ is not contained in $\mathbf{R}_-$. Thus $s_0 - q_1 < s_1 \leqslant s_0$.

Suppose we have already found $s_0, s_1, \dots, s_{j-1}$ such that $s_{j-2} - q_{j-1} < s_{j-1} \leqslant s_{j-2}$. Then we define s_j as the right endpoint of an interval $M_j(k_j) = [r_j, s_j]$ such that $s_j \leqslant s_{j-1}$ and $s_j + q_j > s_{j-1}$. Thus $s_{j-1} - q_j < s_j \leqslant s_{j-1}$, establishing the induction step.

The right endpoints s_j of the intervals $M_j(k_j) = [r_j, s_j] = [s_j - \tau q_j, s_j]$ form a decreasing sequence by construction.

We shall estimate s_n. We have

$$\begin{aligned} s_n &> s_{n-1} - q_n > s_{n-2} - q_{n-1} - q_n > \cdots \\ &> s_1 - q_2 - q_3 - \cdots - q_n \geqslant s_1 - q_2(1 + \sigma + \sigma^2 + \cdots + \sigma^{n-2}) \\ &= s_1 - q_2 \frac{1 - \sigma^{n-1}}{1 - \sigma} \geqslant -q_1 \frac{1 - \sigma^n}{1 - \sigma}. \end{aligned}$$

We observe that $r_1 = s_1 - \tau q_1$, the left endpoint of the first interval $[r_1, s_1]$ lies to the left of s_n if we choose τ properly. Indeed,

$$s_n - r_1 = s_n - s_1 + \tau q_1 > -q_2 \frac{1 - \sigma^{n-1}}{1 - \sigma} + \tau q_1 = \frac{1 - \sigma^{n-1}}{1 - \sigma}(-q_2 + \sigma q_1) \geqslant 0$$

if we choose $\tau = \sigma(1 - \sigma^{n-1})/(1 - \sigma)$.

The point $z = s_n$ is therefore a point where all inequalities

$$b_j e^{(\beta_j + i\beta_j)z} + c_j e^{(\beta_j - i\beta_j)z} \geqslant 2 \cos\theta \, e^{\beta_j z} |b_j| = \cos\theta \, e^{\beta_j z}(|b_j| + |c_j|)$$

hold.

We now have

$$\|v|_I\|_\infty \geqslant \operatorname{Re} v(s_n) = 2 \sum_1^n e^{\beta_j s_n} \operatorname{Re}\left(b_j e^{i\beta_j s_n}\right) \geqslant 2\cos\theta \sum_1^n e^{\beta_j s_n} |b_j| = \cos\theta \|v\|_{\mathscr{V}},$$

where we have used the weights $w_j = e^{\beta_j s_n}$, $j = 1, \ldots, n$.

Remark 15.2. If $\sigma < \frac{1}{3}$ we may choose $\tau = \sigma/(1 - \sigma) < \frac{1}{2}$ and can prove that the left endpoints r_j form an increasing sequence, so that the intervals form a decreasing sequence of sets, and the final interval $[r_n, s_n]$ is contained in all of them.

We may estimate $r_j - r_{j-1}$ as follows.

$$r_j - r_{j-1} = s_j - \tau q_j - s_{j-1} + \tau q_{j-1} > -q_j - \tau q_j + \tau q_{j-1} = \tau q_{j-1} - (1 + \tau) q_j \geqslant 0,$$

since now $\tau = (1 + \tau)\sigma$. This proves that $[r_j, s_j] \subset [r_{j-1}, s_{j-1}]$.

□

Example. Let the longest spatial period be $q_1 = 2000$ meters, corresponding to a temporal period $p_1 = (4\pi\kappa)^{-1} q_1^2 \approx 3.06 \cdot 10^{11}$ seconds ≈ 9699 years. Then with a separation parameter $\sigma = 0.3$ we get $\tau = \sigma/(1 - \sigma) = 3/7 \approx 0.42857$ and $I = [s_n, 0]$ with $s_n \geqslant -1.429 q_1 = -2858$ meters for any number of terms.

16. What remains to be done?

This paper presents just the first steps in an investigation of a model for the reconstruction of past temperatures. A sensitivity analysis should be performed, as well as an investigation of a corresponding discrete model.

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