Duality of convolution operators: A tool for shape analysis

Christer Oscar Kiselman

1. Introduction

Duality is a term which represents a collection of ideas where two sets of mathematical objects confront each other. An important example is the dual of a normed space, where the linear forms on the space operate on its points. A most successful duality is that between the space $\mathscr{D}(\Omega)$ of test functions (smooth functions of compact support) and its dual $\mathscr{D}'(\Omega)$ of distributions. The distributions that are not defined by a locally integrable function live like ghosts in the dark, perceptible only through their actions on test functions.

Typically some but not all properties of an object are preserved under a duality. A function which is nonzero only in a set of Lebesgue measure zero defines the zero distribution. The support function of a set forgets about holes in the set but the information retained enables us to reconstruct the closed convex hull of the set.

Mathematical morphology can be quite helpful in providing guiding concepts and ideas in the study of discrete convolution operators and other topics, like discrete optimization. We shall now apply these ideas to the duality of convolution operators.

Notation

The family of all subsets of a set X is called the ***power set*** of X and will denoted by $\mathscr{P}(X)$. Thus $A \in \mathscr{P}(X)$ if and only if $A \subset X$. This set is ordered by the relation that $A \leqslant B$ if and only if $A \subset B$. We denote by $\mathscr{P}_{\text{finite}}(X)$ the family of all finite subsets of X.

Following Bourbaki (1954: Chapter II, § 5, No. 1, p. 101) we shall denote the set of all mappings from X into Y by $\mathscr{F}(X, Y)$. If G is an abelian semigroup with zero 0, we define the ***support*** of a function $f \colon X \to G$ as the set where f is nonzero. We denote by $\mathscr{F}_{\text{finite}}(X, G)$ the subset of $\mathscr{F}(X, G)$ consisting of all functions which are zero outside a finite subset of X.

We shall write $\mathbf{R}_!$ for the extended real line, obtained by adding two infinities:

$$\mathbf{R}_! = \mathbf{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty].$$

We shall denote by a dot the ***inner product*** of two vectors x and y in $\mathbf{R}^n$: $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. The ***Euclidean norm*** $\|x\|_2$ of a vector x is nonnegative and satisfies $\|x\|_2^2 = x \cdot x$.

2. Counting with infinities

How shall we define a sum like $(+\infty)+(-\infty)$? Can addition $\mathbf{R}\times\mathbf{R} \ni (x, y) \mapsto x+y \in \mathbf{R}$ be extended to an operation $\mathbf{R}_! \times \mathbf{R}_! \ni (x, y) \mapsto x + y \in \mathbf{R}_!$ in reasonable way?

A convenient solution, pioneered by Jean-Jacques Moreau (1970), is to define two extensions, ***upper addition*** and ***lower addition***. The first is an upper semicontinuous mapping from $\mathbf{R}_! \times \mathbf{R}_!$ into $\mathbf{R}_!$; the second a lower semicontinuous mapping. They are denoted by $\dotplus$ and $\underset{\cdot}{+}$ and are defined by the requirements of being commutative and to satisfy

$$\begin{aligned} x \dotplus (+\infty) &= +\infty \text{ for all } x \in \mathbf{R}_!; \\ x \dotplus (-\infty) &= -\infty \text{ for all } x \in [-\infty, +\infty[\,; \text{ and} \\ x \underset{\cdot}{+} y &= -\big((-x) \dotplus (-y)\big) \text{ for all } x, y \in \mathbf{R}_!. \end{aligned} \tag{2.1}$$

3. Minkowski addition and infimal convolution

The Minkowski sum $A + B$ of two subsets can be put into the wider framework of infimal convolution.

Definition 3.1. Given two functions $f, g\colon G \to \mathbf{R}_!$ defined on an abelian semigroup G and with values in the extended real line, we define a new function $h = f \sqcap g$, called the ***infimal convolution*** of f and g, as

$$(f \sqcap g)(z) = h(z) = \inf_{x,y\in G} \Big(f(x) \dotplus g(y);\ x + y = z\Big), \quad z \in G. \tag{3.1}$$ □

The infimum is taken over all elements $x, y \in G$ such that their sum is z, the argument of h. The complication which arises if f takes the value $+\infty$ at x and g takes the value $-\infty$ at y is resolved here by declaring that $+\infty$ shall win—see the definition of upper addition $\dotplus$ in formula (2.1).

We define the ***indicator function*** of a set A, denoted by $\mathbf{indf}_A$, as the function which takes the value 0 in A and $+\infty$ in its complement. We have $\mathbf{indf}_A \sqcap \mathbf{indf}_B = \mathbf{indf}_{A+B}$, showing a generalization of Minkowski addition.

4. Concepts from mathematical morphology

4.1. Ethmomorphisms

Definition 4.1. A relation denoted by $\leqslant$ in a set X is called a ***preorder*** if it is reflexive: $x \leqslant x$ for all $x \in X$, and transitive: for all elements $x, y, z \in X$, if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

A preorder is called an ***order*** if, in addition, it is antisymmetric, i.e., if $x \leqslant y$ and $y \leqslant x$ only if $x = y$.

In preordered spaces the increasing mappings are of importance:

Definition 4.2. If $f\colon X \to Y$ is a mapping from a preordered set X to a preordered set Y, then we say that f is ***increasing*** if for all $x, x' \in X$, the relation $x \leqslant_X x'$ implies $f(x) \leqslant_Y f(x')$.

We shall say that f is ***decreasing*** if for all $x, x' \in X$, the relation $x \leqslant_X x'$ implies $f(x') \leqslant_Y f(x)$.

We shall call a mapping $f\colon X \to Y$ ***coincreasing*** if for all $x, x' \in X$ with $f(x) \leqslant_Y f(x')$, we have $x \leqslant_X x'$. □

The increasing mappings play the same role in the context of preordered sets as the linear mappings in the theory of vector spaces and the continuous mappings in the theory of topological spaces.

A preorder $\leqslant$ is said to be ***finer*** than another preorder $\preccurlyeq$ if the identity mapping $X_{\leqslant} \to X_{\preccurlyeq}$ is increasing.

Definition 4.3. We shall say that a mapping $f\colon X \to X$ is ***idempotent*** if $f \circ f = f$, i.e., if $f(f(x)) = f(x)$ for all $x \in X$. A mapping which is both increasing and idempotent will be called an ***ethmomorphism***.[1] □

4.2. Cleistomorphisms and anoiktomorphisms

Among the ethmomorphisms we shall distinguish those that are either larger than, or smaller than, the identity mapping.

Definition 4.4. A ***cleistomorphism*** in an ordered set X is an ethmomorphism (see Definition 4.3 above) $\kappa\colon X \to X$ which is larger than the identity; in other words, which satisfies the following three conditions.

$$x \leqslant y \text{ implies } \kappa(x) \leqslant \kappa(y), \qquad x, y \in X; \tag{4.1}$$

$$\kappa(\kappa(x)) = \kappa(x), \qquad x \in X; \tag{4.2}$$

$$x \leqslant \kappa(x), \qquad x \in X. \tag{4.3}$$

□

The element $\kappa(x)$ is said to be the ***closure*** of x. Elements x such that $\kappa(x) = x$ are called ***invariant*** for this operator; in the case of cleistomorphisms also ***closed***. An element is closed if and only if it is the closure of some element (and then it is the closure of itself).

Definition 4.5. An ***anoiktomorphism*** in an ordered set X is an ethmomorphism (see Definition 4.3 above) $\alpha\colon X \to X$ which is smaller than the identity; in other words, which satisfies the following three conditions.

$$x \leqslant y \text{ implies } \alpha(x) \leqslant \alpha(y), \qquad x, y \in X; \tag{4.4}$$

$$\alpha(\alpha(x)) = \alpha(x), \qquad x \in X; \tag{4.5}$$

$$\alpha(x) \leqslant x, \qquad x \in X. \tag{4.6}$$

□

The invariant elements are also called ***open*** when we are considering an anoiktomorphism.

[1]The term *ethmomorphism* defined here, as well as the terms *cleistomorphism* and *anoiktomorphism* in the next subsection, were introduced by the author in (2007; 2010), and are formed in analogy with the many terms *homomorphism*, *isomorphism*, *automorphism*, *homeomorphism* etc. Their origins are in the Classical Greek words ἠθμός *(m)* (ethmós) 'strainer, colander'; κλείς *(f)* (kleis) 'key', κλειστός (kleistós) 'closed, that can be shut or closed'; ἄνοιξις *(f)* (ánoixis) 'opening', ανοικτός (anoiktós) 'opened'. Ebbe Vilborg advised me concerning the choice of terms. In English we have other terms of the same origin such as *ethmoid bone*, *cleistogamy*.

4.3. Lattices; upper and lower inverse of a mapping

An ordered set L is said to be a ***complete lattice***, if for any family $(x_j)_{j\in J}$ of elements of L, there is an infimum $z = \bigwedge x_j$ and a supremum $w = \bigvee x_j$. This means that $z \leqslant x_j$ for all $j \in J$ and that, if $z' \leqslant x_j$ for all $j \in J$, then $z' \leqslant z$; and that $w \geqslant x_j$ and that, if $w' \geqslant x_j$ for all $j \in J$, then $w' \geqslant w$. If J has only two elements x_1, x_2, we write $x_1 \wedge x_2$ and $x_1 \vee x_2$.

An ordered set L is said to be a ***lattice*** if the conditions above hold for finite index families J.

In general a mapping does not have an inverse. However, to a mapping defined in a complete lattice, we can define two mappings in the opposite direction that can serve as inverses:

Definition 4.6. Let L be a complete lattice, M a preordered set, and $g\colon L \to M$ any mapping. We then define the ***lower inverse*** $g_{[-1]}\colon M \to L$ and the ***upper inverse*** $g^{[-1]}\colon M \to L$ by

$$g_{[-1]}(y) = \bigvee_{x\in L} \Big(x;\ g(x) \leqslant_M y\Big), \quad y \in M;$$

$$g^{[-1]}(y) = \bigwedge_{x\in L} \Big(x;\ y \leqslant_M g(x)\Big), \quad y \in M. \qquad \square$$

5. Convolution

Definition 5.1. If G is an abelian semigroup we define the ***convolution product*** $h = f * g$ of two functions f, g defined on G and with real or complex values by the formula

$$h(z) = \sum_{\substack{x,y\in G \\ x+y=z}} f(x)g(y), \qquad z \in G, \tag{5.1}$$

provided the sum can be given a meaning. $\square$

If one of f, g is nonzero only in a finite subset of G, the sum is always well defined, but we need to consider more general situations.

The ***Kronecker delta*** δ_a is the function which takes the value 1 at a and is zero elsewhere. For $a = 0$ we get a neutral element: $f * \delta_0 = f$ for all functions f.

It is actually impossible to define a general associative convolution product, as shown by the following simple example.

Example 5.2. Define on $G = \mathbf{Z}$ three functions: $f = 1$ identically; g the difference operator $\delta_0 - \delta_1$; $h =$ the Heaviside function, defined by $h(x) = 0$ for $x \leqslant -1$, $h(x) = 1$ for $x \geqslant 0$. Then $f * g$ and $g * h$ are well defined: $f * g = 0$; $g * h = \delta_0$. It follows that $(f * g) * h = 0$ while $f * (g * h) = f \neq 0$. (We get a warning here: $f * h$ cannot be defined to have finite values—neither can $(f * |g|) * h$, nor $f * (|g| * h)$.) $\square$

Via the Fourier transformation this example is the same as Laurent Schwartz's famous result (1954) that it is impossible to define an associative multiplication for distributions. (We leave aside the question whether there might exist an interesting non-associative convolution algebra.)

If G is an abelian group, we have the well-known inequality

$$\|f * g\|_1 \leqslant \|f\|_1 \cdot \|g\|_1, \qquad f, g \in \mathscr{F}(G, \mathbf{R}), \tag{5.2}$$

making $l^1(G)$, the space of summable functions, into a convolution algebra, and more generally

$$\|f * g\|_p \leqslant \|f\|_1 \cdot \|g\|_p, \qquad f, g \in \mathscr{F}(G, \mathbf{R}), \quad 1 \leqslant p \leqslant \infty. \tag{5.3}$$

If f or g has finite support, commutativity and associativity follow easily. But we will need more general results, with functions that are not summable or even unbounded, just with finite values but possibly growing very fast.

While we cannot hope for a general convolution algebra, we can confront a family of functions with another family, which is an instance of duality. The simplest example of this is that $\mathscr{F}_{\text{finite}}(G, \mathbf{R})$ can work against $\mathscr{F}(G, \mathbf{R})$ and conversely. But there are several other cases of interest.

Provided that we accept $+\infty$ as a value, we can define the convolution product $|f| * |g|$ of the absolute values of any two functions $f, g \in \mathscr{F}(G, \mathbf{R})$. The sum is simply defined to be the supremum of all finite partial sums:

$$(|f| * |g|)(z) = \sup_{A \in \mathscr{P}_{\text{finite}}(G)} \sum_{\substack{x, y \in A \\ x+y=z}} |f(x)| \cdot |g(y)| \leqslant +\infty.$$

Therefore it makes sense to require that $|f| * |g|$ be finite everywhere. If this is so, then also $f * g$ is well defined as a function with finite values; the sum defining it being absolutely convergent at every point. More precisely, we have the following result concerning commutativity and associativity.

Proposition 5.3. *Let G be an abelian group and $f_j \colon G \to \mathbf{R}$, $j = 1, 2, 3$, three functions.*

(1). *If $|f_1| * |f_2|$ is finite everywhere, then $f_1 * f_2$ is defined everywhere as an absolutely converging series, $f_1 * f_2 = f_2 * f_1$, and $|f_1 * f_2| \leqslant |f_1| * |f_2|$.*

(2). *If $(|f_1| * |f_2|) * |f_3|$ is finite everywhere, then $(f_1 * f_2) * f_3$ and $f_1 * (f_2 * f_3)$ are defined everywhere as absolutely converging series, and they are equal.*

First a word about convergence. We shall say that, given a function $F \colon G \to \mathbf{R}$, a series $\sum_{x \in G} F(x)$ ***converges to a sum*** s if for every $\varepsilon > 0$ there exists a finite subset A of G such that, for every finite subset $B \supset A$, we have

$$\left| \sum_{x \in B} F(x) - s \right| \leqslant \varepsilon.$$

We will write this as

$$\lim_{A \to G} \sum_{x \in A} F(x) = s, \text{ or just } \sum_{x \in G} F(x) = s.$$

A well-known necessary and sufficient condition for convergence is ***Cauchy's criterion***:

For every $\varepsilon > 0$ there exists a finite subset A of G such that, for every finite set $C \subset \complement A$, we have $|\sum_{x \in C} F(x)| \leqslant \varepsilon$.

If $|\sum_{x \in C} F(x)| \leqslant \varepsilon$ for all finite sets C which are disjoint from A, then also $\sum_{x \notin A} F(x)$ is well defined and satisfies $|\sum_{x \notin A} F(x)| \leqslant \varepsilon$.

Proof. (1). We denote by g_j the functions defined as f_j in a finite subset A_j of G and as 0 in $G \smallsetminus A_j$, $j = 1, 2, 3$. Clearly $|g_j| \leqslant |f_j|$ and $|g_j| * |g_k| \leqslant |f_j| * |f_k|$.

We know that $g_1 * g_2 = g_2 * g_1$ and that $|f_1| * |f_2| = |f_2| * |f_1|$.

For the first part of the theorem it is easy to see that Cauchy's criterion is satisfied. For any finite subset B of G we have

$$\sum_{y \in B} (f_1(y) - g_1(y)) g_2(x - y) = \sum_{y \in B \smallsetminus A_1} f_1(y) g_2(x - y),$$

which implies that the absolute value of the last sum is majorized by

$$\sum_{y \notin A_1} |f_1(y)| \cdot |f_2(x - y)|,$$

which is smaller than ε if A_1 is large. Hence $\lim_{A_1 \to G}(g_1 * g_2) = f_1 * g_2$.

We go on and find that

$$\sum_{y \in B} f_1(x - y)(f_2(y) - g_2(y)) = \sum_{y \in B \smallsetminus A_2} f_1(x - y) f_2(y),$$

with absolute values

$$\left| \sum_{y \in B \smallsetminus A_2} f_1(x - y)(f_2(y) - g_2(y)) \right| \leqslant \sum_{y \notin A_2} |f_1(x - y)| \cdot |f_2(y)| \leqslant \varepsilon,$$

which shows that $\lim_{A_2 \to G}(f_1 * g_2) = f_1 * f_2$. We conclude that the itererated limit is as we wished for:

$$\lim_{A_2 \to G} \lim_{A_1 \to G} (g_1 * g_2) = f_1 * f_2.$$

Since $g_1 * g_2 = g_2 * g_1$, commutativity follows.

(2). We know that $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$, and also that $(|f_1| * |f_2|) * |f_3| = |f_1| * (|f_2| * |f_3|)$.

Now, since $(|f_1| * |f_2|) * |f_3|$ is finite everywere, we see that also $|f_1| * |f_2|$ is finite everywhere; similarly for all the $|f_j| * |f_k|$. We have now

$$\sum_{y \in B} (f_1 * f_2)(x - y)(f_3(y) - g_3(y)) = \sum_{y \in B \smallsetminus A_3} (f_1 * f_2)(x - y) f_3(y),$$

with absolute values majorized by

$$\sum_{y \notin A_3} |(f_1 * f_2)(x - y)| \cdot |f_3(y)|,$$

which by hypothesis can be arbitrality small when A_3 is large. So the limit $\lim_{A_3 \to G}((f_1 * f_2) * g_3)$ is equal to $(f_1 * f_2) * f_3$, and we conclude for the iterated limit that

$$\lim_{A_3 \to G} \lim_{A_2 \to G} \lim_{A_1 \to G} ((g_1 * g_2) * g_3) = (f_1 * f_2) * f_3.$$

Since the g_j satisfy the associative law, so do the f_j. We are done. □

6. Duality in convolution

Definition 6.1. Given an abelian semigroup G, we define

(6.1) $$\Theta(\mathscr{G},\mathscr{H},\mathscr{K})=\{f\colon G\to\mathbf{R};\ \ \text{for all } g\in\mathscr{G}, |f|*|g|\in\mathscr{H} \text{ and } f*g\in\mathscr{K}\},$$

where $\mathscr{G},\mathscr{H},\mathscr{K}\subset\mathscr{F}(G,\mathbf{R})$, thus defining a transformation

$$\Theta\colon \mathscr{P}(\mathscr{F}(G,\mathbf{R}))\times\mathscr{P}(\mathscr{F}(G,\mathbf{R}))\times\mathscr{P}(\mathscr{F}(G,\mathbf{R}))\to\mathscr{P}(\mathscr{F}(G,\mathbf{R})).$$ □

Clearly $\Theta(\mathscr{G},\mathscr{H},\mathscr{K})$ is decreasing in $\mathscr{G}$ and increasing in $\mathscr{H}$ and $\mathscr{K}$.

Three particular cases will be of interest:

Definition 6.2. If we take both $\mathscr{H}$ and $\mathscr{K}$ equal to $\mathscr{F}(G,\mathbf{R})$, then

(6.2) $$\Theta(\mathscr{G},\mathscr{H},\mathscr{K})=\Gamma(\mathscr{G})=\{f\in\mathscr{F}(G,\mathbf{R});\ \ \text{for all } g\in\mathscr{G}, |f|*|g|<+\infty\},$$

where the formula defines a mapping $\Gamma\colon \mathscr{P}(\mathscr{F}(G,\mathbf{R}))\to\mathscr{P}(\mathscr{F}(G,\mathbf{R}))$. □

Definition 6.3. If $\mathscr{H}=\mathscr{F}(G,\mathbf{R})$ and $\mathscr{K}=\{f\in\mathscr{F}(G,\mathbf{R});\ f\geqslant 0\}$, then $\Theta(\mathscr{G},\mathscr{H},\mathscr{K})$ is equal to

(6.3) $$\Gamma^{\geqslant 0}(\mathscr{G})=\{f\in\mathscr{F}(G,\mathbf{R});\ \ \text{for all } g\in\mathscr{G}, |f|*|g|<+\infty \text{ and } f*g\geqslant 0\},$$

where the formula defines $\Gamma^{\geqslant 0}\colon \mathscr{P}(\mathscr{F}(G,\mathbf{R}))\to\mathscr{P}(\mathscr{F}(G,\mathbf{R}))$. □

Definition 6.4. If $\mathscr{H}=\mathscr{F}(G,\mathbf{R})$ and $\mathscr{K}=\{f\in\mathscr{F}(G,\mathbf{R});\ f=0\}$, then $\Theta(\mathscr{G},\mathscr{H},\mathscr{K})$ is equal to

(6.4) $$\Gamma^{0}(\mathscr{G})=\{f\in\mathscr{F}(G,\mathbf{R});\ \ \text{for all } g\in\mathscr{G}, |f|*|g|<+\infty \text{ and } f*g=0\},$$

where the formula defines $\Gamma^{0}\colon \mathscr{P}(\mathscr{F}(G,\mathbf{R}))\to\mathscr{P}(\mathscr{F}(G,\mathbf{R}))$. □

Given two preordered sets X and Y, a ***Galois correspondence*** is a pair (F,G) of mappings, $F\colon X\to Y$ and $G\colon Y\to X$ which are both decreasing and such that the two compositions $G\circ F\colon X\to X$ and $F\circ G\colon Y\to Y$ are larger than the identity mappings. It follows that $G\circ F$ and $F\circ G$ are cleistomorphisms.

The Galois correspondences can be subsumed under the concept of lower and upper inverses: If we provide Y with the opposite preorder, then both F and G are increasing and the lower inverse of F is equal to G; the lower inverse of G is equal to F.

Clearly the pairs (Γ,Γ), $(\Gamma^{\geqslant 0},\Gamma^{\geqslant 0})$, and (Γ^{0},Γ^{0}) are all Galois correspondences.

We should determine all classes of functions which are fixed points for the corresponding cleistomorphisms.

When studying convolution equations $\mu*u=f$ with μ of finite support, it is convenient to introduce two variants of $\Gamma^{\geqslant 0}$ in (6.3):

(6.5) $$\Gamma_0^{\geqslant 0}(\mathscr{G})=\{u\in\mathscr{F}(G,\mathbf{R});\ \ \text{for all } \mu\in\mathscr{G}, \mu*u\geqslant 0\},\qquad \mathscr{G}\subset\mathscr{F}_{\text{finite}}(G,\mathbf{R});$$

(6.6) $$\Gamma_1^{\geqslant 0}(\mathscr{G})=\{\mu\in\mathscr{F}_{\text{finite}}(G,\mathbf{R});\ \ \text{for all } u\in\mathscr{G}, \mu*u\geqslant 0\},\qquad \mathscr{G}\subset\mathscr{F}(G,\mathbf{R}).$$

The first is just the restriction of $\Gamma^{\geqslant 0}$ to the set of functions with finite support. Now $(\Gamma_0^{\geqslant 0},\Gamma_1^{\geqslant 0})$ is a Galois correspondence.

Example 6.5. Let us define $\mathscr{G}_0 = \mathscr{F}_{\mathrm{finite}}(G, \mathbf{R})$ and $\mathscr{G}_1 = \mathscr{F}(G, \mathbf{R})$. Then $\Gamma(\mathscr{G}_0) = \mathscr{G}_1$ and $\Gamma(\mathscr{G}_1) = \mathscr{G}_0$; it follows that $(\Gamma^{\circ 2})(\mathscr{G}_j) = \mathscr{G}_j$, $j = 0, 1$. So both $\mathscr{G}_0$ and $\mathscr{G}_1$ are fixed points for $\Gamma^{\circ 2}$. □

Several other types of duality may be studied.

We note that $\Gamma(\mathscr{G})$ is a vector subspace of $\mathscr{F}(G, \mathbf{R})$ which contains $\mathscr{F}_{\mathrm{finite}}(G, \mathbf{R})$. Also

$$\mathscr{G} \subset \mathscr{G} + \mathscr{F}_{\mathrm{finite}}(G, \mathbf{R}) \subset \Gamma^{\circ 2}(\mathscr{G}),$$

which implies that

$$\Gamma(\mathscr{G}) = \Gamma(\mathscr{G} + \mathscr{F}_{\mathrm{finite}}(G, \mathbf{R})) = \Gamma(\Gamma^{\circ 2}(\mathscr{G})) = \Gamma^{\circ 3}(\mathscr{G}).$$

This implies that Γ can be defined in the quotient space $\mathscr{F}(G, \mathbf{R})/\mathscr{F}_{\mathrm{finite}}(G, \mathbf{R})$, a space analogous to the space of singularities $\mathscr{D}'(\Omega)/\mathscr{E}(\Omega)$, the space of distributions modulo the subspace of smooth functions. Also the more elementary singularity space $C^0(\Omega)/C^\infty(\Omega)$ is interesting.

Example 6.6. Let $\mathscr{G}$ be the subfamily of $\mathscr{F}(G, \mathbf{R})$ whose only member is the constant function with the value 1. Then $\Gamma(\mathscr{G}) = l^1(G)$, the set of functions $g\colon G \to \mathbf{R}$ such that $\sum_{x \in G} |g(x)|$ is finite. Moreover $(\Gamma \circ \Gamma)(\mathscr{G}) = l^\infty(G)$, the family of all bounded functions $h\colon G \to \mathbf{R}$. The higher powers $\Gamma^{\circ k}(\mathscr{G})$ are alternatively equal to $l^1(G)$ (odd $k \geqslant 1$) and $l^\infty(G)$ (even $k \geqslant 2$). So both $l^1(G)$ and $l^\infty(G)$ are fixed points for $\Gamma \circ \Gamma$. □

7. Concepts from convexity theory

A subset A of $\mathbf{R}^n$ is said to be ***convex*** if

$$\{a, b\} \subset A \text{ implies that } [a, b] \subset A,$$

where $[a, b]$ is the ***segment*** with endpoints a and b:

$$[a, b] = \{(1 - t)a + tb;\ 0 \leqslant t \leqslant 1\}.$$

A function $f\colon \mathbf{R}^n \to \mathbf{R}_!$ is said to be ***convex*** if its finite epigraph

$$\mathbf{epi}^{\mathrm{finite}}(f) = \{(x, t) \in \mathbf{R}^n \times \mathbf{R};\ f(x) \leqslant t\}$$

is convex.

The ***convex hull*** of a set A, denoted by $\mathbf{cvxh}(A)$, is the smallest convex set which contains A. The ***convex envelope*** of a function $f\colon A \to \mathbf{R}_!$, denoted by $\mathbf{cvxe}(f)$, is the largest convex function $F\colon \mathbf{R}^n \to \mathbf{R}_!$ such that $F(x) \leqslant f(x)$ for all $x \in A$.

A function f with real values can be written as $f = f^+ - f^-$, where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

Definition 7.1. A ***Jensen function*** is a function $\mu\colon \mathbf{R}^n \to \mathbf{R}$ of the form

$$\mu = \sum_{j=1}^{N} \lambda_j \delta_{b(j)} - \delta_a, \tag{7.1}$$

where the scalars λ_j and the points $b^{(j)}$ satisfy the restrictions

$$\lambda_j > 0, \quad j = 1, \cdots, N, \qquad \sum_{j=1}^{N} \lambda_j = 1, \text{ and } \sum_{j=1}^{N} \lambda_j b^{(j)} = a. \tag{7.2}$$

The set of all Jensen functions will be denoted by $\mathscr{J}(\mathbf{R}^n)$.

We shall say that $\mu\colon \mathbf{R}^n \to \mathbf{R}$ is a ***generalized Jensen function*** if

(α'). μ is nonzero at finitely many points only;

(β'). $\sum_{x\in\mathbf{R}^n} \mu(x) = 0$;

(γ'). $\sum_{x\in\mathbf{R}^n} \mu(x)x = 0$;

(δ'). $\sum_{x\in\mathbf{R}^n} \mu(x)g^+(x) \geqslant 0$ for all functions g of the form $g(x) = \gamma \cdot x + c$, where γ is a vector in $\mathbf{R}^n$ and c a real constant.

We shall denote the set of all generalized Jensen functions by $\mathscr{J}_{\mathrm{gen}}(\mathbf{R}^n)$. □

Here (β') means that the mass of μ^+ shall be equal to the mass of μ^-.

We define the ***barycenter***, to be denoted by $\mathbf{bary}(f)$, of a nonnegative function $f \in \mathscr{F}_{\mathrm{finite}}(\mathbf{R}^n, \mathbf{R})$ which is not identically zero as

$$\mathbf{bary}(f) = \frac{\sum_{x\in\mathbf{R}^n} f(x)x}{\sum_{x\in\mathbf{R}^n} f(x)} \in \mathbf{R}^n.$$

Thus (γ') says that μ^+ and μ^- shall have the same barycenter if $\mu \neq 0$.

It is well known that a function $f\colon \mathbf{R}^n \to \mathbf{R}$ is convex if and only if $\mu * f \geqslant 0$ for all Jensen functions μ.

Clearly $\mathscr{J}(\mathbf{R}^n) \subset \mathscr{J}_{\mathrm{gen}}(\mathbf{R}^n)$.

Given $f\colon A \to \mathbf{R}_!$, where A is a subset of $\mathbf{R}^n$, its convex envelope $(\mathbf{cvxe}(f))(a)$ is the infimum of all expressions $(f * \mu)(a)$ where

$$\mu = \sum_{j=1}^{N} \lambda_j \delta_{b^{(j)}},$$

when the λ_j and the $b^{(j)}$ vary under the restrictions (7.2).

A function $f\colon A \to \mathbf{R}_!$ is said to be ***convex extensible*** if it is the restriction of a convex function defined in all of $\mathbf{R}^n$.

8. Duality in convexity theory: the Fenchel transformation

Given any function $f\colon \mathbf{R}^n \to \mathbf{R}_!$ we define its ***Fenchel transform*** $\tilde{f}$ by

$$\tilde{f}(\xi) = \sup_{x\in\mathbf{R}^n} (\xi \cdot x - f(x)), \qquad \xi \in \mathbf{R}^n. \tag{8.1}$$

The function $\widetilde{f}$ is convex, lower semicontinuous, and takes the value $-\infty$ only if it is identically equal to $-\infty$. The second Fenchel transform $\widetilde{\widetilde{f}}$ is the largest minorant of f with these three properties. We have $\widetilde{\widetilde{f}} = f$ if and only if f itself has the three mentioned properties.

A special case is when $f = \mathbf{indf}_A$. Then $\widetilde{f} = H_A$, the support function of A, defined by

$$H_A(\xi) = \sup_{x \in A} \xi \cdot x, \qquad \xi \in \mathbf{R}^n. \tag{8.2}$$

Starting from the support function we can form the set

$$B = \{x \in \mathbf{R}^n;\ \text{for all } \xi \in \mathbf{R}^n,\ \xi \cdot x \leqslant H_A(\xi)\}. \tag{8.3}$$

The Hahn–Banach theorem shows that B is the closed convex hull of A.

Thus from knowledge of the support function we can reconstruct the closed convex hull of A, but we lose information concerning holes in A.

Remark 8.1. By embedding a set into a space of higher dimension, we can reconstruct its closure from its support function, thus with a much smaller loss of information. To be precise, given A in $\mathbf{R}^n$ we can define a function $f\colon \mathbf{R}^n \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{1}{2}\|x\|_2^2 \text{ when } x \in A; \\ +\infty \text{ when } x \notin A. \end{cases}$$

So A is lifted to the paraboloid in $\mathbf{R}^n \times \mathbf{R}$ of equation $t = \frac{1}{2}\|x\|_2^2$. Then the second Fenchel transform $\widetilde{\widetilde{f}}(x)$ is equal to $\frac{1}{2}\|x\|_2^2 f$ if and only if x belongs to the closure of A. This is also visible in the support function of the set $\Big\{(x,t);\ t \geqslant \widetilde{\widetilde{f}}(x)\Big\}$. See also my paper (1981). □

9. Representations in terms of elementary convex functions

The most elementary convex functions are the affine ones, and the Fenchel transformation shows that certain convex functions can be represented as a supremum of affine functions, viz.,

$$\widetilde{\widetilde{f}}(x) = \sup_{\xi \in \mathbf{R}^n} (\xi \cdot x - \widetilde{f}(\xi)), \qquad x \in \mathbf{R}^n.$$

So the affine functions $x \mapsto h(x) = \xi \cdot x + c$, where γ is a vector in $\mathbf{R}^n$ and c a real constant, are the building blocks of the convex function $\widetilde{\widetilde{f}}$; we know the exact conditions under which the latter equals f. This is in analogy with Fourier synthesis, where a function is represented as a sum or integral of simple oscillations $x \mapsto \mathrm{e}^{\mathrm{i}\xi \cdot x}$, $\xi \in \mathbf{R}^n$.

However, in relation to convolution it would be of interest to build up a convex function as a sum rather than a supremum of elementary convex functions. Of course a sum of affine functions is itself affine, so we must go one step further:

Definition 9.1. Let us say that a function g on $\mathbf{R}^n$ is an ***elementary convex function*** if it has the form $g = (h_1 \vee h_2)|_{\mathbf{Z}^n}$ for some affine functions $h_1, h_2\colon \mathbf{R}^n \to \mathbf{R}$. □

Since $h_1 \vee h_2 = h_1 + (h_2 - h_1)^+$, it is equivalent to use affine functions and their positive part.

The question is now whether all convex extensible functions on $\mathbf{Z}^n$ can be written as the restriction of a sum of elementary convex functions.

Example 9.2. The function given by $f(x) = |x_1 + x_2| \vee |x_1 - x_2|$, $x \in \mathbf{Z}^2$, can be written as a sum g of four elementary convex functions: $g(x) = x_1^+ + x_1^- + x_2^+ + x_2^-$. □

Example 9.3. Let now $f(x) = \|x\|_2^2$, $x \in \mathbf{Z}^2$. The affine function h_a, $a \in \mathbf{Z}^2$, defined by

$$h_a(x) = \|a\|_2^2 + (2a_1 + 1)(x_1 - a_1) + (2a_2 + 1)(x_2 - a_2), \qquad x \in \mathbf{R}^2,$$

agrees with $\mathbf{cvxe}(f)$ in the square

$$P_a = [a_1, a_1 + 1] \times [a_2, a_2 + 1] \subset \mathbf{R}^2, \qquad a \in \mathbf{Z}^2.$$

We note that $h(x) = \lambda_1 x_1 + \lambda_2 x_2$ is a maximal affine minorant of $\mathbf{cvxe}(f)$ for all λ_j satisfying $|\lambda_j| \leqslant 1$, but not all of these agree with $\mathbf{cvxe}(f)$ in a maximal set.

We now measure the distance from the origin to a point a in $\mathbf{Z}^2$ by the infinity norm $\|a\|_\infty = \max |a_j|$ and define

$$g(x) = h_0 + \sum_{k=1}^{\infty} \sum_{\substack{a \in \mathbf{Z}^2 \\ \|a\|_\infty = k-1}} \sum_{\substack{b \in \mathbf{Z}^2 \\ \|b\|_\infty = k}} (h_b - h_a)^+, \qquad x \in \mathbf{Z}^2, \tag{9.1}$$

a denumerable sum of elementary convex functions. We have $g|_{\mathbf{Z}^2} = f$. □

Let us say that a function $f\colon \mathbf{Z}^n \to \mathbf{R}_!$ is of ***fast growth*** if

$$\liminf_{\|x\|_2 \to +\infty} \frac{f(x)}{\|x\|_2} = +\infty.$$

This property implies that the graph of the convex envelope of f consists of bounded polytopes in $\mathbf{R}^n \times \mathbf{R}$.

Theorem 9.4. *Given a convex extensible function $f\colon \mathbf{Z}^n \to \mathbf{R}$ of fast growth, there are denumerably many elementary convex functions $g_j\colon \mathbf{R}^n \to \mathbf{R}$, $j \in \mathbf{Z}$, such that $f(x) = \sum_{j \in \mathbf{Z}} g_j(x)$, $x \in \mathbf{Z}^n$.*

Proof. This result is easy to prove for $n = 1$ and less so for $n \geqslant 2$. Let us therefore take the case of one dimension first.

Given $f\colon \mathbf{Z} \to \mathbf{R}$ we define $h_j(x) = f(j) + \lambda_j(x - j)$, $x \in \mathbf{R}$, where $\lambda_j = f(j+1) - f(j)$. Thus h_j is an affine function taking the same value as f at $x = j$ and $x = j + 1$. We note that $\lambda_j \leqslant \lambda_{j+1}$ and that $(h_{j+1}(x) - h_j(x))^+ = 0$ for $j, x \in \mathbf{Z}$ satisfying $j \geqslant x - 1$.

Then

$$g = \sum_{j=-\infty}^{-1} (h_j - h_{j+1})^+ + h_0 + \sum_{j=1}^{\infty} (h_j - h_{j-1})^+$$

is a sum of elementary convex functions taking the same values as f at all integer points. Indeed, for a given $x \in \mathbf{N}$, the first sum is zero and the last sum goes only over $j = 0, \ldots, x-1$, yielding a telescoping series with sum equal to $h_x(x) - h_0(x)$, implying that $g(x) = h_x(x) = f(x)$ since we add $h_0(x)$. Similarly, for $x \in \mathbf{Z}$, $x \leqslant 0$, the last sum is zero and the first goes only over $j = x, \ldots, -1$, resulting again in $g(x) = f(x)$.

For $n \geqslant 2$, we have to construct a distance like we did in Example 9.3. This will be done by means of a neighborhood relation.

We note that the convex envelope of f is a supremum of denumerably many maximal affine functions h_j, $j \in \mathbf{N}$. If f is not the restriction of an affine function, it has in fact non-denumerably many maximal affine minorants, but we shall restrict attention to those that are maximal not only as to their values, but also in the sense that the set of all $x \in \mathbf{R}^n$ where the affine minorant agrees with $\mathbf{cvxe}(f)$ is maximal. It cannot then be contained in a hyperplane and must have a nonempty interior.

We define a polytope

$$P_j = \{x \in \mathbf{R}^n;\ h_j(x) = \sup_{i\in\mathbf{N}} h_i(x)\}, \qquad j \in \mathbf{N},$$

thus a convex set of nonempty interior. Then we define a neighborhood relation $\diamond$ in $\mathbf{N}$ by declaring that $j \diamond k$ if and only if $j \neq k$ and $P_j \cap P_k \neq$ ‰. We start with an arbitrary polytope P_0 and define a distance from 0 by saying that the distance between $a \in \mathbf{N}$ and 0 is equal to k if there are indices $t_0, t_1, \dots, t_k$ such that $t_0 = 0$ and $t_k = a$ and $t_i \diamond t_{i+1}$, $i = 0, \dots, k-1$, but there is no shorter chain. Then the index set M_k, $k \in \mathbf{N}$, is defined to be the set of indices with distance k to 0. So $\mathbf{N}$ is a disjoint union of all the index sets M_k. With this notation we define

$$g = h_0 + \sum_{k\in\mathbf{N}} \sum_{j\in M_k} \sum_{i\in M_{k-1}} (h_j - h_i)^+,$$

where we define $M_{-1} =$ ‰. It follows that $g = \sup_{j\in\mathbf{N}} h_j = \mathbf{cvxe}(f)$. Thus $g|_{\mathbf{Z}^n} = f$. □

10. Some examples of duality

We may ask for descriptions of $\Gamma^0(\mathscr{G})$, defined by (6.4), for any given class of functions $\mathscr{G}$. Also of interest is the second power $(\Gamma^0)^{\circ 2} = \Gamma^0 \circ \Gamma^0$ of Γ^0, the cleistomorphism associated to Γ^0.

Similarly we can study $\Gamma^{\geqslant 0}(\mathscr{G})$ and $(\Gamma^{\geqslant 0})^{\circ 2}(\mathscr{G})$ defined by (6.3).

10.1. Functions of one variable

In particular we shall take $\mathscr{G}$ as a singleton, $\mathscr{G} = \{\beta\}$ for some $\beta \in \mathscr{F}_{\text{finite}}(\mathbf{Z}, \mathbf{R})$. We may always assume that $\inf(x;\ \beta(x) \neq 0)$ is equal to 0, and that $\beta = \delta_0 - \gamma$ for some $\gamma \in \mathscr{F}(\mathbf{Z}, \mathbf{R})$ which vanishes for all points $x \leqslant 0$. Then γ^{*j} has its support in $[jq, +\infty[\ \subset [j, +\infty[$ if the support of γ is contained in $[q, +\infty[$, $q \geqslant 1$. A convolution inverse to β is given by

$$\beta^{*(-1)} = (\delta_0 - \gamma)^{*(-1)} = \delta_0 + \gamma + \gamma * \gamma + \gamma * \gamma * \gamma + \cdots = \sum_{j=0}^{\infty} \gamma^{*j}.$$

It is the only inverse of β which is zero for large negative arguments. The series converges nicely, but $\beta^{*(-1)}$ usually does not have finite support. Given any function μ, we may ask whether $\beta^{*(-1)} * \mu$ has finite support.

We obtain

$$\Gamma^0(\{\beta\}) = \{f \in \mathscr{F}(\mathbf{Z}, \mathbf{R});\ \beta * f = 0\} = \{f;\ \gamma * f = f\},$$

which is a kind of periodicity for f.

Also

$$(\Gamma^0)^{\circ 2}(\{\beta\}) = \Big\{\mu \in \mathscr{F}(\mathbf{Z},\mathbf{R});\ \text{such that, for all } f \in \mathscr{F}(\mathbf{Z},\mathbf{R}) \text{ satisfying } |\mu| * |f| < +\infty,\ |\beta| * |f| < +\infty,\ \text{and } \beta * f = 0,\ \text{we have } \mu * f = 0\Big\}.$$

For brevity, let us write ρ for $\sum_{j=1}^{\infty} \gamma^{*j}$, so that the inverse of $\delta_0 - \gamma$ is $\delta_0 + \rho$. Let now $\mu \in \mathscr{F}_{\text{finite}}(\mathbf{Z},\mathbf{R})$ be given. In order for $\beta^{*(-1)} * \mu = \mu + \rho * \mu$ to have finite support, it is necessary and sufficient that $(\rho * \mu)(x)$ vanish for large positive values of x. The functions ρ that can occur are those which have a certain periodicity property: $\rho * \gamma = \rho - \gamma$.

If $(\rho * \mu)(x) = 0$ for large values of x, then $\mu = \beta * \nu$ for some ν with finite support, and this implies that $\Gamma^0(\{\mu\})$ contains $\Gamma^0(\{\beta\})$. In general I find it difficult to say more now in this general framework, but for special choices of β we can analyze the situation completely.

Let $Y_a \colon \mathbf{Z} \to \mathbf{Z}$ be the ***Heaviside function*** which is zero for $x \leqslant a - 1$ and one for $x \geqslant a$. Then $Y_a * (\delta_0 - \delta_1) = \delta_a$. So we can solve the equation $(\delta_0 - \delta_1) * u = f$ for an f which vanishes for large negative arguments by taking $u = Y_0 * f$. We also define

$$V_a(x) = x - a + 1, \qquad x, a \in \mathbf{Z}, \tag{10.1}$$

the restriction of an affine function, and note that $V_a^+ * (\delta_0 - \delta_1) = Y_a$, thus $V_a^+ * (\delta_0 - \delta_1)^{*2} = \delta_a$.

The convolution product $Y_0 * Y_0$ is an inverse of $(\delta_0 - \delta_1)^{*2}$; we have $(Y_0 * Y_0)(x) = V_0^+(x)$, where $V_0(x) = x + 1$, $x \in \mathbf{Z}$.

Example 10.1. A first example is $\beta = \delta_0 - \delta_1$, $\gamma = \delta_1$. Then $(\beta * f)(x) = f(x) - f(x-1)$, and $\Gamma^0(\{\beta\})$ is equal to the set of all constant functions.

For $((\Gamma^0)^{\circ 2})(\{\delta_0 - \delta_1\})$ the following three properties are equivalent.

(10.1.1). $\mu \in (\Gamma^0)^{\circ 2}(\{\beta\})$;

(10.1.2). $\mu * 1 = 0$;

(10.1.3). $\mu = \beta * \nu$ for some $\nu \in \mathscr{F}_{\text{finite}}(\mathbf{Z},\mathbf{R})$.

To see that the second property implies the third we note that, for any given $\mu \in \mathscr{F}_{\text{finite}}(\mathbf{Z},\mathbf{R})$,

$$(\beta^{*(-1)} * \mu)(x) = (Y_0 * \mu)(x) = \sum_{y=0}^{\infty} \mu(x-y),$$

a sum which is equal to $(\mu * 1)(x)$ for large positive values of x, and which vanishes for large x if and only if $\mu * 1 = 0$, i.e., if and only if μ vanishes on all constants. □

Example 10.2. Second, we take $\beta = (\delta_0 - \delta_1)^{*2}$, implying that $\gamma = \delta_0 - \beta = 2\delta_1 - \delta_2$. Then $(\beta * f)(x) = f(x) - 2f(x-1) + f(x-2)$, implying that the functions f which satisfy $\beta * f = 0$ are the restrictions to $\mathbf{Z}$ of the affine functions, and that $\Gamma^0(\{\beta\})$ is equal to the set of all restrictions of affine functions.

For $(\Gamma^0)^{\circ 2}(\{\beta\})$ the following three properties are equivalent.

(10.2.1). $\mu \in (\Gamma^0)^{\circ 2}(\{\beta\})$;

(10.2.2). $\mu * f = 0$ for all functions f which are restrictions to $\mathbf{Z}$ of an affine function;

(10.2.3). $\mu = \beta * \nu$ for some $\nu \in \mathscr{F}_{\text{finite}}(\mathbf{Z}, \mathbf{R})$.

To see that the second property implies the third, we first note that $V_0^+ = \beta^{*(-1)}$ and that (10.2.2) implies that $\mu * V_0 = 0$ for the function V_0 defined in (10.1) above; $V_0(x) = x + 1$. For any function $\mu \in \mathscr{F}_{\text{finite}}(\mathbf{Z}, \mathbf{R})$,

$$(\beta^{*(-1)} * \mu)(x) = (V_0^+ * \mu)(x) = \sum_{y=0}^{\infty} V_0^+(x-y)\mu(y), \qquad x \in \mathbf{Z},$$

a sum which is equal to $(V_0 * \mu)(x)$ for large positive values of x and vanishes for large negative values of x. Thus $(\beta^{*(-1)} * \mu) - (V_0 * \mu) \cdot Y_0$ has finite support. We denote it by ν, so that

$$\beta^{*(-1)} * \mu = (V_0 * \mu) \cdot Y_0 + \nu$$

and, if $V_0 * \mu = 0$,

$$\mu = \beta * \beta^{*(-1)} * \mu = \beta * \nu.$$

So property (10.2.2) implies property (10.2.3). □

Example 10.3. Third, we take again $\beta = (\delta_0 - \delta_1)^{*2}$ and now ask for $\Gamma_0^{\geqslant 0}(\{\beta\})$ and $\Gamma_1^{\geqslant 0}(\Gamma_0^{\geqslant 0}(\{\beta\}))$. The first set is the set $CVX(\mathbf{R}, \mathbf{R})|_{\mathbf{Z}}$ of all convex extensible functions, and the second is the set of all functions μ of the form $\beta * \sigma$ where σ has finite support and satisfies $\sigma \geqslant 0$. Since $\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0}$ is idempotent, we conclude that

$$(\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0})^{\circ k}(\beta) = (\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0})(\beta), \qquad k = 1, 2, \ldots,$$

and

$$\Gamma_0^{\geqslant 0} \circ (\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0})^{\circ k}(\beta) = CVX(\mathbf{R}, \mathbf{R})|_{\mathbf{Z}}, \qquad k = 0, 1, \ldots.$$

More precisely, let us consider the following three conditions.

(10.3.1). $\mu \in \Gamma_1^{\geqslant 0}(\Gamma_0^{\geqslant 0}(\{\beta\}))$.

(10.3.2). $\mu = \beta * \sigma$ for some $\sigma \in \mathscr{F}_{\text{finite}}(\mathbf{Z}, \mathbf{R})$ with only nonnegative values.

(10.3.3). μ is a finite positive sum of translations of β: $\mu = \sum_{j \in \mathbf{Z}} \lambda_j (\beta * \delta_j)$, with $\lambda_j \geqslant 0$ and only finitely many of them nonzero.

That (10.3.2) and (10.3.3) are equivalent is easy to see. That (10.3.2) implies (10.3.1) is also easy: If (10.3.2) holds we obtain

$$\mu * f = (\beta * \sigma) * f = \sigma * (\beta * f) \geqslant 0$$

for all convex extensible functions f, so μ belongs to $\Gamma_1^{\geqslant 0}(\Gamma_0^{\geqslant 0}(\{\beta\}))$, i.e., (10.3.1) holds.

Conversely, suppose now that (10.3.1) holds. We define σ as the function which vanishes for large negative values of the argument and is such that $\beta * \sigma = \mu$. Then, since $\mu * V_a^+ \geqslant 0$ for the elementary convex extensible functions $V_a^+(x) = (x - a + 1)^+$, where $V_a(x) = x - a + 1$ as in (10.1), we obtain

$$0 \leqslant (\mu * V_a^+)(x) = ((\beta * \sigma) * V_a^+)(x) = (\sigma * (\beta * V_a^+))(x) = (\sigma * \delta_a)(x) = \sigma(x - a).$$

(Note that the associativity formula $(\beta * \sigma) * V_a^+ = \sigma * (\beta * V_a^+)$ holds, since both factors σ and V_a^+ vanish for large negative arguments.)

So clearly $\sigma \geqslant 0$ everywhere; secondly, if $x - a$ is a large positive number, then

$$(\mu * V_a^+)(x) = \sum_{y \leqslant x-a} \mu(y)(x-y-a+1) = \sum_{y \in \mathbf{Z}} \mu(y)(x-y-a+1)$$

vanishes; consequently, so does σ for large positive arguments. This proves that σ has finite support. □

10.2. Duality for functions of several variables

Proposition 10.4. $\Gamma_0^{\geqslant 0}(\mathscr{J}(\mathbf{R}^n)) = CVX(\mathbf{R}^n)|_{\mathbf{Z}^n}$, *where* $\mathscr{J}(\mathbf{R}^n)$ *is as in Definition 7.1.*

Proof. This is just the well-known statement that a function $f\colon \mathbf{Z}^n \to \mathbf{R}$ satisfies $\mu * f \geqslant 0$ for all Jensen functions μ if and only if is has a convex extension. □

Proposition 10.5. $(\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0})(\mathscr{J}(\mathbf{R}^n)) = \mathscr{J}_{\mathrm{gen}}(\mathbf{R}^n)$. *Continuing, we have*

$$(\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0})^{\circ k} = \Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0} \quad \textit{and} \quad \Gamma_0^{\geqslant 0} \circ (\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0})^{\circ k} = \Gamma_0^{\geqslant}, \quad k = 1, 2, \dots .$$

Proof. By definition, $(\Gamma_1^{\geqslant 0} \circ \Gamma_0^{\geqslant 0})(\mathscr{J}(\mathbf{R}^n))$ is equal to

$$\Big\{\mu \in \mathscr{F}_{\mathrm{finite}}(\mathbf{Z}^n, \mathbf{R});\ \mu * f \geqslant 0 \text{ for all } f \in CVX(\mathbf{R}^n)|_{\mathbf{Z}^n}\Big\}.$$

If μ belongs to $\Gamma_1^{\geqslant 0}(CVX(\mathbf{R}^n)|_{\mathbf{Z}^n})$, then $\mu * f \geqslant 0$ for all functions f in $CVX(\mathbf{R}^n)$, in particular if f is an elementary convex function: the conditions (α'), (β'), (γ') and (δ') in Definition 7.1 are satisfied. Thus $\Gamma_1^{\geqslant 0}(CVX(\mathbf{R}^n)|_{\mathbf{Z}^n}) \subset \mathscr{J}_{\mathrm{gen}}(\mathbf{R}^n)$.

In the other direction, assume now that μ belongs to $\mathscr{J}_{\mathrm{gen}}(\mathbf{R}^n)$. Then $\mu * g \geqslant 0$ for every elementary convex function, and therefore also $\mu * f \geqslant 0$ if f is a finite sum of elementary convex functions. We can now apply Theorem 9.4 to the function f_ε defined as $f_\varepsilon(x) = f(x) + \varepsilon \|x\|_2^2$, with $\varepsilon > 0$, a function of fast growth, to prove that $\mu * f_\varepsilon \geqslant 0$ even if f is a denumerably infinite sum of elementary convex functions: at every given point x, the convolution $(\mu * f_\varepsilon)(x)$ agrees with the convolution of μ with a finite partial sum of the infinite sum of elementary convex functions. Letting ε tend to zero we see that also $\mu * f \geqslant 0$. □

11. Duality between classes of functions and second-order difference operators

We define the difference operator D_a by $(D_a f)(x) = f(x+a) - f(x)$.

The idea is to go in two directions: fix a set of difference operators and consider the set of all functions that are convex with respect to these, and, conversely, fix a class of functions and consider the set of difference operators for which this class satisfies the inequality $D_b D_a f \geqslant 0$.

Given an abelian semigroup G, we define two mappings, special cases of the mappings $\Gamma_1^{\geqslant 0}$ and $\Gamma_0^{\geqslant 0}$ defined in Section 6:

$$\Phi\colon \mathscr{P}(G \times G) \to \mathscr{P}(\mathscr{F}(G, \mathbf{R})) \text{ and } \Psi\colon \mathscr{P}(\mathscr{F}(G, \mathbf{R})) \to \mathscr{P}(G \times G)$$

by

(11.1) $\Phi(A) = \Big\{ f \in \mathscr{F}(G, \mathbf{R});\ D_b D_a f \geqslant 0 \text{ for all } (a,b) \in A \Big\}, \qquad A \in \mathscr{P}(G \times G),$

and

(11.2) $\Psi(\mathscr{G}) = \Big\{ (a,b) \in G \times G;\ D_b D_a f \geqslant 0 \text{ for all } f \in \mathscr{G} \Big\}, \qquad \mathscr{G} \in \mathscr{P}(\mathscr{F}(G, \mathbf{R})).$

This is just the operations $\Gamma_1^{\geqslant 0}$ and $\Gamma_0^{\geqslant 0}$ with elements of the special form

$$\mu = \delta_0 - \delta_{-a} - \delta_{-b} + \delta_{-a-b}$$

and indexed by the pair (a,b): $\mu * f = D_b D_a f$. The definitions are motivated by the fact that these special convolution operators play an important role for discrete convexity, in particular for the study of convexity of marginal functions, which we now come to.

11.1. Duality applied to marginal functions

Given a function of two groups of variables, $F\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}_!$, we define its ***marginal function*** H in $\mathbf{R}^n$ as

$$H(x) = \inf_{y \in \mathbf{R}^m} F(x,y), \qquad x \in \mathbf{R}^n.$$

That H is convex if F is convex is a simple fact. However, it is not trivial to extend this result to discrete situations.

Let now $\mathscr{G}$ be a set of functions g defined on $\mathbf{Z} \times \mathbf{Z}$ and let $h(x) = \inf_{y \in \mathbf{Z}} g(x,y)$ be the marginal function of g. Define $\mathscr{H}$ as the set of all such marginal functions obtained from $g \in \mathscr{G}$.

Then, given a set $\mathscr{G}$ of functions on $\mathbf{Z} \times \mathbf{Z}$, we may consider the set $\Psi(\mathscr{G})$ of all pairs $(a,b) \in \mathbf{Z}^2 \times \mathbf{Z}^2$ such that $D_b D_a g \geqslant 0$ for all $g \in \mathscr{G}$ and also the set $\Psi(\mathscr{H})$ of all pairs $(a,b) \in \mathbf{Z} \times \mathbf{Z}$ such that $D_b D_a h \geqslant 0$ for all $h \in \mathscr{H}$, the set of marginal functions obtained from functions in $\mathscr{G}$. We can then study the relation between $\Psi(\mathscr{G})$ and $\Psi(\mathscr{H})$.

In Kiselman & Samieinia (2010, 2017) we have obtained results on marginal functions in a discrete setting, but these can be generalized using the described duality. We can go up to higher dimensions, and we can study other discrete subsets of $\mathbf{R}^n \times \mathbf{R}^m$ than $\mathbf{Z}^n \times \mathbf{Z}^m$.

12. Discrete convexity defined by convolution

Let us take up again the definition of $\Gamma_0^{\geqslant 0}$, defined by equation (6.5). We shall say that a function $f \in \mathscr{F}(G, \mathbf{R})$ is ***$\mathscr{M}$-positive*** if it satisfies the inequality $f * \mu \geqslant 0$ for all $\mu \in \mathscr{M}$. Here $\mathscr{M}$ can be any subset of $\mathscr{F}_{\text{finite}}(G, \mathbf{R})$, including the case of a singleton subset $\{\mu\}$. Thus the definition is an instance of duality: the set of all $\mathscr{M}$-positive functions is equal to $\Gamma_0^{\geqslant 0}(\mathscr{M})$.

Example 12.1. The simplest example is when $\mu = \delta_0$. Then f is $\{\mu\}$-positive if and only if $f \geqslant 0$. A more interesting guiding example is when is $G = \mathbf{Z}$ and $\mu = \frac{1}{2}\delta_{-1} - \delta_0 + \frac{1}{2}\delta_1$. Then $f * \mu \geqslant 0$ if and only if f is convex extendible. □

We know from Section 7 that a nonzero function μ satisfies $\mu * h = 0$ for all affine functions h on $\mathbf{R}^n$ if and only if μ^+ and μ^- have the same mass and the same barycenter.

Convex functions defined on $\mathbf{R}^n$ have the property that $f \vee g$ is convex if both f and g are. This would be a desirable property, and we can characterize the classes $\mathscr{M}$ which have this property. We first need the following lemma.

Lemma 12.2. *Let $\mu \in \mathscr{F}_{\mathrm{finite}}(\mathbf{R}^n, \mathbf{R})$ be a nonzero function. Then there is a natural number m such that $\mu * p = 0$ for all polynomials p of order at most $m-1$ and a polynomial q of order m such that $\mu * q$ is the constant* 1.

Proof. If $\mu * p = 0$ for all polynomials we shall prove that μ is zero. The Fourier–Laplace transform of μ is

$$\hat{\mu}(\zeta) = \sum_{x \in \mathbf{R}^n} \mu(x) \mathrm{e}^{-\mathrm{i}\zeta \cdot x}, \qquad \zeta \in \mathbf{C}^n,$$

an entire function. We differentiate it and get

$$\frac{\partial^{k_1}}{\partial \zeta^{k_1}} \cdots \frac{\partial^{k_n}}{\partial \zeta^{k_n}} \hat{\mu}(\zeta) = \sum_{x \in \mathbf{R}^n} \mu(x) \mathrm{e}^{-\mathrm{i}\zeta \cdot x} (-\mathrm{i})^{\|k\|_1} x_1^{k_1} \cdots x_n^{k_n}, \qquad \zeta \in \mathbf{C}^n, \; k \in \mathbf{N}^n.$$

So if $\mu * p = 0$ for all monomials $p(x) = x^k = x_1^{k_1} \cdots x_n^{k_n}$, then all derivatives of $\hat{\mu}$ at the origin vanishes, which implies that $\hat{\mu}$ is zero; thus also μ. This proves that for every nonzero μ there is an integer m such that $\mu * p$ vanishes for polynomials of order $< m$ but not for all polynomials of order m. This also implies that $\mu * q$ is a constant for every polynomial of order m, since lower-order terms in q do not influence the value of the convolution product. The constant is nonzero for some choice of q and can be taken to be 1. □

Theorem 12.3. *Let $\mathscr{M}$ be any subset of $\mathscr{F}_{\mathrm{finite}}(\mathbf{R}^n, \mathbf{R})$, the elements of which are nonzero and annihilate all affine functions, thus satisfying the conditions (β') and (γ') in Definition 7.1. Suppose that* supp μ^- *is a singleton set for all elements μ of $\mathscr{M}$. If $(f_j)_{j \in J}$ is any family of $\mathscr{M}$-positive functions, then also $f = \sup_{j \in J} f_j$ is $\mathscr{M}$-positive.*

Conversely, if μ is a nonzero function which annihilates all affine functions and is such that $f \vee g$ is $\{\mu\}$-positive for all $\{\mu\}$-positive functions f and g, then the support of μ^- is a singleton.

Proof. For every $j \in J$ we have

$$\mu^- * f_j \leqslant \mu^+ * f_j \leqslant \mu^+ * f.$$

We now take the supremum over all j and obtain

$$\sup_j (\mu^- * f_j) \leqslant \mu^+ * f.$$

The general inequality $\mu^- * f \geqslant \sup_j (\mu^- * f_j)$ is an equality when the support of μ^- is a singleton set. So we have

$$\mu^- * f = \sup_j (\mu^- * f_j) \leqslant \sup_j (\mu^+ * f_j) \leqslant \mu^+ * f,$$

proving that f is $\mathscr{M}$-positive.

For the converse we take m and q according to Lemma 12.2. (We note that m must be at least 2 under the hypotheses, but can be arbitrarily large.)

Let a and $b \neq a$ be such that $\mu(a) < 0$ and $\mu(b) < 0$. Without loss of generality we may suppose that $\mu(a) = \min \mu$. Take

$$f = \delta_{-a} + Aq \text{ and } g = \delta_{-b} + Aq,$$

where q is the polynomial we get from Lemma 12.2 and A is a constant to be determined. Then

$$f \vee g = (\delta_{-a} \vee \delta_{-b}) + Aq = \delta_{-a} + \delta_{-b} + Aq.$$

We have

$$(\mu * f)(x) = \mu(a + x) + A, \quad (\mu * g)(x) = \mu(b + x) + A,$$

as well as

$$(\mu * (f \vee g))(x) = \mu(a + x) + \mu(b + x) + A, \qquad x \in \mathbf{R}^n.$$

To prove the proposition it suffices to have

$$\mu(a + x) \geqslant -A, \quad \mu(b + x) \geqslant -A \text{ for every } x \in \mathbf{R}^n \text{ and } \mu(a) + \mu(b) < -A.$$

We now choose $A = -\mu(a) = -\min \mu$. The inequalities then become

$$\mu(a + x) \geqslant \mu(a), \quad \mu(b + x) \geqslant \mu(a) \text{ for every } x \in \mathbf{R}^n \text{ and } \mu(a) + \mu(b) < \mu(a),$$

which are all true. □

Adama Arouna Koné's doctoral dissertation (2016) contains a result like Theorem 12.3 but where the converse is stated and proved under two special hypotheses, viz. that the minimum of μ is attained at one point only, and that $m = 2$ (Koné 2014:79, Proposition 4.3.11). Here these hypotheses have been eliminated.

References

Bourbaki, N[icolas]. 1954. *Éléments de mathématique. Théorie des ensembles.* Chapitre I et II. Paris: Hermann & C[ie].

Kiselman, Christer O. 1981. How to recognize supports from the growth of functional transforms in real and complex analysis. **In:** Machado, Silvio, Ed., *Functional Analysis, Holomorphy, and Approximation Theory* (Proceedings of the Seminário de Analíse Funcional, Holomorfia e Teoria da Aproximação, Universidade Federal do Rio de Janeiro, Brazil, August 7–11, 1978), pp. 366–372. Lecture Notes in Mathematics 843. Berlin et al.: Springer-Verlag.

Kiselman, Christer O. 2007. Division of mappings between complete lattices. **In:** Banon, G. J. F.; Barrera, J.; de Mendonça Braga-Neto, U., Eds. *Mathematical Morphology and its Applications to Signal and Image Processing. Proceedings of the 8th International Symposium on Mathematical Morphology, Rio de Janeiro, RJ, Brasil, October 10–13, 2007*, pp. 27–38. São José dos Campos, SP, Brasil: MCT/INPE.

Kiselman, Christer O. 2010. Inverses and quotients of mappings between ordered sets. *Image and Vision Computing* **28**, 1429–1442.

Kiselman, Christer O.; Samieinia, Shiva. 2010. Convexity of marginal functions in the discrete case. **In:** Samieinia (2010, paper VI).

Kiselman, Christer O.; Samieinia, Shiva. 2017. Convexity of marginal functions in the discrete case. **In:** Andersson, Mats; Boman, Jan; Kiselman, Christer; Sigurdsson, Ragnar, Eds. *Analysis Meets Geometry. A Tribute to Mikael Passare*, pp. 287–309. Basel: Birkhäuser.

Koné, Adama Arouna. 2016. *Géométrie digitale utilisée pour la discrétisation et le recouvrement optimal des objects euclidiens.* Doctoral Thesis presented and approved on 2016 January 14 at l'Université des Sciences, des Techniques et des Technologies de Bamako (USTTB). vi + 114 pp.

Moreau, J.-J. 1970. Inf-convolution, sous-additivité, convexité des fonctions numériques, *J. Math. Pures et Appl.* **49**, 109–154.

Samieinia, Shiva. 2010. *Digital Geometry, Combinatorics, and Discrete Optimization.* Doctoral Thesis published 2010 December 15 and defended 2011 January 21. Stockholm: Stockholm University, Department of Mathematics.

Schwartz, Laurent. 1954. Sur l'impossibilité de la multiplication des distributions. *C. R. Acad. Sci. Paris* **239**, 847–848.

Uppsala University, Department of Information Technology
Paper address: P. O. Box 337, SE-751 05 Uppsala
Amber addresses: `kiselman@it.uu.se`, `christer@kiselman.eu`
URL: `www.cb.uu.se/~kiselman`

Submitted on 2019 April 30.
Two referee reports received on 2020 July 20.
A revised version sent on 2020 July 23.
Accepted for publication on 2021 July 14.