
ELSEVIER
Volume 28, issue 10, October 2010 ISSN 0262-8856
Special Issue:
ISMM 2007
image
AND
vision
COMPUTING
Available online at www.sciencedirect.com
ScienceDirect

Image and Vision Computing 28 (2010) 1429–1442

Contents lists available at ScienceDirect

Image and Vision Computing

journal homepage: www.elsevier.com/locate/imavis

Inverses and quotients of mappings between ordered sets

Christer O. Kiselman *

Uppsala University, Department of Mathematics, P.O. Box 480, SE-751 06 Uppsala, Sweden

ARTICLE INFO

Article history:
Received 11 September 2008
Received in revised form 14 April 2009
Accepted 15 June 2009

Keywords:
Preordered set
Complete lattice
Generalized inverse of a mapping
Division of mappings
Epigraph
Hypograph
Galois connection
Residuated mapping
Adjunction
Dilation
Erosion
Ethmomorphism
Morphological filter
Cleistomorphism
Closure operator
Anoiktomorphism
Kernel operator

ABSTRACT

In this paper, we study inverses and quotients of mappings between ordered sets, in particular between complete lattices, which are analogous to inverses and quotients of positive numbers. We investigate to what extent a generalized inverse can serve as a left inverse and as a right inverse, and how an inverse of an inverse relates to the identity mapping. The generalized inverses and quotients are then used to create a convenient formalism for dilations and erosions as well as for cleistomorphisms (closure operators) and anoiktomorphisms (kernel operators).

1. Introduction

Lattice theory is a mature mathematical theory thanks to the pioneering work by Garrett Birkhoff, Øystein Ore and others in the first half of the twentieth century. A standard reference is still Birkhoff's book [3], first published in 1940. Developments in lattice theory originate in several branches of mathematics, for instance algebra [6,5], logic [42,15], general topology and functional analysis [15, p.xxx–xxxii], convexity theory [40] and, most important as a background for this paper, mathematical morphology with applications in image processing (books by Matheron [28], Serra [35,38], and Heijmans [17]; articles by Heijmans and Ronse [20], Ronse [32], Ronse and Heijmans [33,34], and Serra [39]). Other areas where concepts from lattice theory are used include semantics (abstract interpretation) of programming, the theory of fuzzy sets, fuzzy logic, and formal concept analysis [13]. For general lattice theory a standard reference is Grätzer [16].

This variety of sources for fundamental concepts has led to varying terminology and hence to difficulties in tracing history.

In this paper, we shall study inverses and quotients of mappings between ordered sets which are analogous to inverses $1/y$ and quotients x/y of positive numbers. The theory of lower and upper inverses defined in Section 3 generalizes the theory of Galois connections as well as residuation theory and the theory of adjunctions. We investigate in Section 6 to what extent a generalized inverse can serve as a left inverse and as a right inverse, and how an inverse of an inverse relates to the identity mapping. The generalized inverses and quotients are then used in Section 9 to create a convenient formalism for a unified treatment of dilations $\delta : L \to M$ and erosions $\varepsilon : L \to M$ as well as of cleistomorphisms (closure operators) $\kappa : L \to L$ and anoiktomorphisms (kernel operators) $\alpha : L \to L$.

Often we require of the ordered sets studied that they shall be complete lattices. However, of the various phenomena brought together here, the Galois connections are the oldest, and they make

* Tel.: +46 184713216; fax: +46 184713201.
E-mail addresses: kiselman@math.uu.se, christer@kiselman.eu
URL: http://www.math.uu.se/~kiselman.

0262-8856/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.imavis.2009.06.014

sense between preordered sets which are not necessarily complete lattices or even lattices. A goal will therefore be to study this general situation and relate it to the more special theories of residuation and adjunction.

Both inverses and quotients come in two versions, lower and upper. It turns out that anoiktomorphisms can be characterized as lower quotients of the form $f/_{\star}f$, and cleistomorphisms as upper quotients $f/^{\star}f$.

To define an inverse of a general mapping seems to be a hopeless task. However, if the mapping is between preordered sets, there is some hope of constructing mappings that can serve in certain contexts just like inverses do. This is our task here.

Part of the results of the present paper were reported in my conference contribution (2007), however without proofs and with fewer examples. My lectures in the Spring Semesters of 2002 and 2004 also contained some of the results; see (2002a).

2. Definitions

Definition 2.1. A *preorder* in a set X is a binary relation which is reflexive (for all $x \in X, x \leqslant x$) and transitive (for all $x, y, z \in X$, $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$). An *order* is a preorder which is antisymmetric (for all $x, y \in X, x \leqslant y$ and $y \leqslant x$ imply $x = y$).

Birkhoff [2, p. 7] uses *quasi-ordered system*; in (1995:20) *quasi-ordering* and *quasi-ordered set*. Other terms are *preordering*, *quasiorder*, and *pseudoordering*. Nowadays *preorder* is more common (e.g., Gierz et al. [15, p. 1]).

To any preorder $\leqslant$ we introduce an equivalence relation $x \sim y$ defined as $x \leqslant y$ and $y \leqslant x$. If $\leqslant$ is an order, this equivalence relation is just equality. If we have two preorders, we say that $\leqslant_1$ is *stronger than* or *finer than* $\leqslant_2$ if for all x and y, $x \leqslant_1 y$ implies $x \leqslant_2 y$. We also say that $\leqslant_2$ is *weaker than* or *coarser than* $\leqslant_1$. The finest preorder is the *discrete order*, defined as equality; the coarsest preorder is the *chaotic preorder* given by $x \leqslant y$ for all x and y.

Definition 2.2. Given a preordered set $(X, \leqslant)$, we define the *opposite preordered set* $(X, \geqslant)$ as the set X equipped with the opposite preorder. We shall write X^{op} for this preordered set. Thus $x \leqslant_{X^{\mathrm{op}}} y$ if and only if $y \leqslant_X x$.

Definition 2.3. Given a mapping $f : X \to (Y, \leqslant)$ of a set into a preordered set $(Y, \leqslant)$, we define a mapping $f^{\mathrm{op}} : X \to (Y, \geqslant)$ taking the same values as f; given a mapping $f : (X, \leqslant) \to Y$ of a preordered set $(X, \leqslant)$ into a set Y, we define $f_{\mathrm{op}} : (X, \geqslant) \to Y$ taking the same values; finally, if $f : (X, \leqslant) \to (Y, \leqslant)$ is a mapping between preordered sets, we define $f^{\mathrm{op}}_{\mathrm{op}} = (f^{\mathrm{op}})_{\mathrm{op}} = (f_{\mathrm{op}})^{\mathrm{op}} : (X, \geqslant) \to (Y, \geqslant)$. For brevity we shall also write these mappings as

$$f^{\mathrm{op}} : X \to Y^{\mathrm{op}}, \; f_{\mathrm{op}} : X^{\mathrm{op}} \to Y, \quad \text{and } f^{\mathrm{op}}_{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}.$$

We note that $(f^{\mathrm{op}})^{\mathrm{op}} = f$ and $(f_{\mathrm{op}})_{\mathrm{op}} = f$ whenever defined.

Definition 2.4. A *complete lattice* is an ordered set such that any family $(x_j)_{j \in J}$ of elements possesses a smallest majorant and a largest minorant. We denote them by $\bigvee_{j \in J} x_j$ and $\bigwedge_{j \in J} x_j$, respectively.

A complete lattice must possess a smallest element, to be denoted by $\mathbf{0}$, and a largest element, $\mathbf{1}$.

Definition 2.5. If $f : X \to Y$ is a mapping of a set into another, we define its *graph* as the set

$$\operatorname{graph} f = \{(x, y) \in X \times Y; y = f(x)\}.$$

If Y is preordered, we define also its *epigraph* and its *hypograph* as

$$\operatorname{epi} f = \{(x,y) \in X \times Y; f(x) \leqslant y\}, \;\; \operatorname{hypo} f = \{(x,y) \in X \times Y; y \leqslant f(x)\}.$$

We shall also need the *strict epigraph* and the *strict hypograph*,

$$\operatorname{epi}_s f = \{(x,y) \in X \times Y; f(x) < y\}, \;\; \operatorname{hypo}_s f = \{(x,y) \in X \times Y; y < f(x)\},$$

of a function $f : X \to Y$, where $a < b$ means that $a \leqslant b$ and $a \neq b$.

Obviously $\operatorname{epi} f = \operatorname{hypo} f^{\mathrm{op}}$.

If X and Y are given, any mapping $X \to Y$ is determined by its graph, and, if Y is an ordered set, also by its epigraph as well as by its hypograph. It is often convenient to express properties of mappings in terms of their epigraphs or hypographs; for examples, see Proposition 4.3 and formulas (5.4).

Definition 2.6. If two preordered sets X and Y and a mapping $f : X \to Y$ are given, we shall say that f is *increasing* if

for all $x, x' \in X$, $x \leqslant_X x' \Rightarrow f(x) \leqslant_Y f(x')$,

and that f is *coincreasing* if

for all $x, x' \in X$, $f(x) \leqslant_Y f(x') \Rightarrow x \leqslant_X x'$.

Finally f is said to be *decreasing* or *codecreasing* if f^{op} (equivalently f_{op}) is increasing or coincreasing, respectively.

If f is increasing, then so is $f^{\mathrm{op}}_{\mathrm{op}}$, whereas f^{op} and f_{op} are decreasing.

The terms *increasing* and *decreasing* are widely used. Birkhoff [3, p. 2], Blyth and Janowitz [6, p. 6], Blyth [5, p. 5], and Grätzer [16, p. 20] call an increasing mapping *order-preserving* or *isotone*. Blyth and Janowitz [6, p. 6] and Blyth [5, p. 5] call a decreasing mapping *order-inverting* or *antitone*. Gierz et al. used *order-preserving* and *monotone* (2003, p. 5) as well as *antitone* (2003, p. 35).

The term *coincreasing* appears in my lecture notes (2002a, p. 12).

To emphasize the symmetry between the two notions, we define, given any mapping $f : X \to Y$ between preordered sets, a preorder $\leqslant_f$ in X by the requirement that $x \leqslant_f x'$ if and only if $f(x) \leqslant_Y f(x')$. Then f is increasing if and only if $\leqslant_X$ is finer than $\leqslant_f$, and f is coincreasing if and only if $\leqslant_X$ is coarser than $\leqslant_f$.

A comparison with topology is in order here. If $f : X \to Y$ is a mapping of a topological space X into a topological space Y with topologies (families of open sets) τ_X and τ_Y, we can define a new topology τ_f in X as the family of all sets $\{x \in X; f(x) \in V\}, V \in \tau_Y$. Then f is continuous if and only if τ_X is finer than τ_f.

Definition 2.7. A mapping $f : L \to M$ of a complete lattice L into a complete lattice M is said to be a *dilation* if $f(\bigvee_{j \in J} x_j) = \bigvee_{j \in J} f(x_j)$ for all families $(x_j)_{j \in J}$ of elements in L.

A mapping is said to be an *erosion* if $f^{\mathrm{op}}_{\mathrm{op}}$ is a dilation, i.e., if $f(\bigwedge_{j \in J} x_j) = \bigwedge_{j \in J} f(x_j)$ for all families $(x_j)_{j \in J}$.

A mapping is said to be an *anti-erosion* if f^{op} is an erosion, i.e., if $f(\bigwedge_{j \in J} x_j) = \bigvee_{j \in J} f(x_j)$ for all families $(x_j)_{j \in J}$.

A mapping is said to be an *anti-dilation* if f^{op} is a dilation, i.e., if $f(\bigvee_{j \in J} x_j) = \bigwedge_{j \in J} f(x_j)$ for all families $(x_j)_{j \in J}$.

We note that a dilation must satisfy $f(\mathbf{0}_L) = \mathbf{0}_M$, an erosion $f(\mathbf{1}_L) = \mathbf{1}_M$.

Matheron in his pioneering treatise (1975:17) used the terms *dilatation* and *erosion* for operations $\mathscr{P}(\mathbf{R}^n) \to \mathscr{P}(\mathbf{R}^n)$. Serra [36,38] defined dilations and erosions as here in the case of complete lattices with $L = M$; anti-erosions and anti-dilations were introduced by Serra [37].

Singer [40, p.172] uses the term *duality* for an anti-erosion. The study of dualities in the sense of Singer is therefore equivalent to that of dilations or erosions.

An explanation for the terms *dilation* and *erosion* is furnished by the operations on subsets of an abelian group G:

$$\delta(A) = A + S, \quad \varepsilon(B) = \{x; x + S \subset B\}, \quad A, B \in \mathscr{P}(G),$$

where S is a fixed subset of G, called the *structuring element*. In typical cases, like taking S as a disk in $\mathbf{R}^2$, δ dilates the image A and ε erodes it in a natural sense. The equivalence

$$\delta(A) \subset B \iff A \subset \varepsilon(B), \quad A, B \in \mathscr{P}(G),$$

is the very basis for the passing from $\mathscr{P}(\mathbf{R}^n)$ or $\mathscr{P}(\mathbf{Z}^n)$ to general complete lattices. We shall see that the lower inverse of δ is ε, and that the upper inverse of ε is δ.

Definition 2.8. A mapping $\eta : X \to X$ of a preordered set X into itself is said to be an *ethmomorphism* if it is increasing and idempotent. If in addition it is extensive, i.e., $\eta(x) \geqslant x$ for all $x \in X$, then it is said to be a *cleistomorphism*; if it is antiextensive, i.e., $\eta(x) \leqslant x$ for all $x \in X$, then it is called an *anoiktomorphism*.[1]

Ethmomorphisms are of central importance in image processing but elusive and difficult to treat. The special cases cleistomorphisms and anoiktomorphisms are much easier to handle.

For the notions just defined many terms have been used. Other terms for ethmomorphism are *morphological filter*[2] (Serra [38, p. 104]), *projection operator* and *projection* (Gierz et al. [15, p. 26]). For cleistomorphism other terms include *closure operation* (Ore [30, p. 494]), *closure mapping* (Blyth and Janowitz [6, p. 9]), *closing* (Matheron [28, p. 187]; Serra [35, p. 56]), *hull operator* (Singer [40, p. 8]), *closure operator* (Gierz et al. [15, p. 26]); in French *fermeture de Moore* (Dubreil and Dubreil-Jacotin [11, p. 177]) and *application enveloppante* (Kiselman [22, p. 336]; term proposed by André Hirschowitz). For anoiktomorphism there are several other terms: *dual closure mapping* (Blyth and Janowitz 6, p. 9]), *opening* (Matheron [28, p. 187]; Serra [35, p. 56]), *kernel operator* (Gierz et al. [15, p. 26]).

3. Inverses of mappings

In general a mapping $g : X \to Y$ between sets does not have an inverse. If g is injective, we may define a left inverse $u : Y \to X$, thus with $u \circ g = \mathbf{id}_X$, where $\mathbf{id}_X$ denotes the identity mapping in X, defining $u(y)$ in an arbitrary way when y is not in the image of g. If g is surjective, we may define a right inverse $v : Y \to X$, thus with $g \circ v = \mathbf{id}_Y$. We then need to define $v(y)$ as an element of the preimage $\{x; g(x) = y\}$. In the general situation this has to be done using the axiom of choice. In a complete lattice, however, it could be interesting to define $v(y)$ as the supremum or infimum of all x such that $g(x) = y$, even though this supremum or infimum need not belong to the set. At any rate, the preimage of y is contained in the interval defined by the infimum and the supremum. However, for various purposes it is convenient to take instead the infimum of all x such that $g(x) \geqslant y$ or the supremum of all x such that $g(x) \leqslant y$. This yields better monotonicity properties. (We include the case $g(x) = y$ by letting the preorder in Y be the discrete preorder.) We make the following definitions, where we refrain from assuming that X is a complete lattice.

Definition 3.1. Let X be an ordered set, Y a preordered set, and $g : X \to Y$ any mapping. We then define the *lower inverse* $g_{[-1]} : D(g_{[-1]}) \to X$ as the mapping

$$g_{[-1]}(y) = \bigvee_{x \in X}(x; g(x) \leqslant_Y y) = \bigvee_{x \in X}(x; (x, y) \in \text{epi } g), \quad y \in D(g_{[-1]}). \tag{3.1}$$

The mapping is defined for all $y \in Y$ such that the supremum exists in X; the set of all such y constitutes the domain of definition of $g_{[-1]}$, denoted by $D(g_{[-1]})$.

We define the *upper inverse*, denoted $g^{[-1]}$, as the mapping $((g^{\text{op}}_{\text{op}})_{[-1]})^{\text{op}}_{\text{op}}$, i.e.,

$$g^{[-1]}(y) = \bigwedge_{x \in X}(x; g(x) \geqslant_Y y) = \bigwedge_{x \in X}(x; (x, y) \in \text{hypo } g), \quad y \in D(g^{[-1]}). \tag{3.2}$$

Here the mapping $g^{\text{op}}_{\text{op}}$ is defined in Definition 2.3.

We also define mappings $((g^{\text{op}})_{[-1]})_{\text{op}} = ((g_{\text{op}})^{[-1]})^{\text{op}}$ and $((g^{\text{op}})^{[-1]})_{\text{op}} = ((g_{\text{op}})_{[-1]})^{\text{op}}$:

$$((g^{\text{op}})_{[-1]})_{\text{op}}(y) = \bigvee_{x \in X}(x; g(x) \geqslant_Y y), \quad y \in D(((g^{\text{op}})_{[-1]})_{\text{op}}); \tag{3.3}$$

$$((g^{\text{op}})^{[-1]})_{\text{op}}(y) = \bigwedge_{x \in X}(x; g(x) \leqslant_Y y), \quad y \in D(((g^{\text{op}})^{[-1]}))_{\text{op}}. \tag{3.4}$$

There are mainly two situations when the generalized inverses are defined in all of Y. The first is when X is a complete lattice. The second is when for example epi $g = (\text{hypo } h)^{\smallsmile}$ for some mapping $h : Y \to X$, where we define, following Birkhoff [3, p.3],

$$R^{\smallsmile} = \{(y, x) \in Y \times X; (x, y) \in R\} \tag{3.5}$$

for any subset R of $X \times Y$. Then $\{x \in X; g(x) \leqslant_Y y\} = \{x \in X; x \leqslant_X h(y)\}$, showing that the supremum of the elements in the set exists and is equal to $h(y)$. In fact, this is the case when (h, g) is an adjunction as we shall see (Section 4.3). Thus $g_{[-1]} = h$ is defined everywhere; $D(g_{[-1]}) = Y$.

In the case of complete lattices X and Y, these four generalized inverses were introduced by Banon and Barrera [1, p. 311], who used them to construct decompositions of general mappings between complete lattices.[3] They denoted them as follows:

$$g_{[-1]} = \overline{g}\cdot, \quad g^{[-1]} = \cdot\underline{g}, \quad ((g^{\text{op}})_{[-1]})_{\text{op}} = \cdot\overline{g}, \quad ((g^{\text{op}})^{[-1]})_{\text{op}} = \underline{g}\cdot.$$

In my notes (2002a) and my paper (2007), I studied only the first two, assuming X to be a complete lattice and Y to be a preordered set.

As a first observation, let us note that $g_{[-1]}$ and $g^{[-1]}$ are always increasing, while $((g^{\text{op}})_{[-1]})_{\text{op}}$ and $((g^{\text{op}})^{[-1]})_{\text{op}}$ are decreasing. If X and Y possess largest elements $\mathbf{1}_X$ and $\mathbf{1}_Y$, then $g_{[-1]}(\mathbf{1}_Y) = \mathbf{1}_X$. Similarly, if there are smallest elements $\mathbf{0}_X$ and $\mathbf{0}_Y$, then $g^{[-1]}(\mathbf{0}_Y) = \mathbf{0}_X$. If Y has the chaotic preorder, then both inverses are constant, $g_{[-1]} = \mathbf{1}_X$ and $g^{[-1]} = \mathbf{0}_X$ identically.

We note that we always have

$$\text{epi } g \cap (X \times D(g_{[-1]})) \subset (\text{hypo } g_{[-1]})^{\smallsmile}, \tag{3.6}$$

in other words, if $g_{[-1]}(y)$ is defined and $g(x) \leqslant y$, then $g_{[-1]}(y) \geqslant x$; and

$$\text{hypo } g \cap (X \times D(g^{[-1]})) \subset (\text{epi} g^{[-1]})^{\smallsmile}, \tag{3.7}$$

in other words, if $g^{[-1]}(y)$ is defined and $g(x) \geqslant y$, then $g^{[-1]}(y) \leqslant x$. Here $R^{\smallsmile}$ for a subset R of $X \times Y$ is defined by (3.5). In general these inclusions are strict as we shall see below.

Note that we do not require in (3.1) that the set of all x such that $g(x) \leqslant_Y y$ shall have a largest element. In other words, the supremum in (3.1), even if it exists, is not necessarily a maximum; see Examples 3.3, 5.1, and 5.2. Using terms from Blyth [5, p. 26], the set of all x such that $g(x) \leqslant y$ is not necessarily an ideal in X, although it is a down-set if g is increasing. Similarly, the infimum in (3.2) is not necessarily a minimum; the set of all x such that $g(x) \geqslant y$ is not necessarily a filter in X, although it is an up-set if g is increasing.

The special situation when the supremum in (3.1) is actually a maximum is equivalent to the inequality $g(g_{[-1]}(y)) \leqslant_Y y$; if this

[1] Cf. the noun *ēthmós* 'strainer' and the adjectives *kleistós* 'closed' and *anoiktós* 'open' in Classical Greek. I am grateful to Ebbe Vilborg for help with these words.

[2] Not to be abbreviated to "filter": a *filter* in a lattice L is a subset F of L such that $x \wedge y$ and $x \vee z$ belong to F whenever $x, y \in F$ and $z \in L$.

[3] I am grateful to Gerald Banon for pointing this out to me; I was unaware of their paper when I wrote my paper (2007).

holds for all $y \in D(g_{[-1]})$, we have $g \circ g_{[-1]} \leqslant_Y \mathbf{id}_Y|_{D(g_{[-1]})}$. This can happen whether g is increasing or not. Except for the simplest lattices there exist increasing mappings which do not have this property:

Proposition 3.2. *Let L be a complete lattice which is not totally ordered, and let M be a complete lattice with at least two elements. Then there exists an increasing mapping $g : L \to M$ such that $g(g_{[-1]}(\mathbf{0}_M)) \not\leqslant \mathbf{0}_M$.*

Proof. Let a, b be any two non-comparable elements of L and define $g(x) = \mathbf{1}_M$ if $x \geqslant a \vee b$; $g(x) = \mathbf{0}_M$ otherwise. (In particular $g(a) = g(b) = \mathbf{0}_M$.) Then $g_{[-1]}(y) \geqslant a \vee b$ for all $y \in M$, so that $g(g_{[-1]}(y)) = \mathbf{1}_M$ for all y, in particular $g(g_{[-1]})(\mathbf{0}_M) = \mathbf{1}_M \not\leqslant \mathbf{0}_M$. □

Example 3.3. Take $X = Y = [0, 1]$ and $g : [0, 1] \to [0, 1]$. Define $x_0 = g_{[-1]}(0) = \sup(x; f(x) \leqslant 0)$. The inequality $g(g_{[-1]}(0)) \leqslant 0$ means that $g(x_0) = 0$, which need not be the case, even if the supremum is attained and g is increasing, for instance if $g = \chi_{[1/2,1]}$, the characteristic function of the interval $[1/2, 1]$.

4. Special cases of inverses of mappings

The inverses we have defined in Section 3 generalize a situation which has been known for a long time, although under different names and with different fields of applications. Our results generalize the theory of Galois connections, equivalently residuation theory and the theory of adjunctions, to a more general situation, a situation which appears even in very simple cases like Proposition 3.2 and Example 3.3; we shall see other examples in the next section. In complete lattices the upper and lower inverses always exist, whereas only very special mappings are upper or lower adjoints.

It seems that this generalization of residuation theory has not been considered in the contexts of the branches of mathematics mentioned in Subsections 4.1, 4.2 and 4.3 below. However, Singer [40, p. 176] defines the *dual* $M \to L$ of a mapping $g : L \to M$, which, after a change of order in L, is the lower inverse defined here. He notes the inclusion corresponding to (3.6) and proves that it is an equality if and only if g is a duality (anti-erosion).

Let us now briefly recall three classical definitions.

4.1. Galois connections

If we require both that $g \circ g_{[-1]} \leqslant \mathbf{id}_Y$ shall hold and that g shall be increasing, then we are in a situation which has been studied a long time.

Definition 4.1. Given two preordered sets X and Y, a *Galois connection* is a pair (F, G) of decreasing mappings $F : X \to Y$, $G : Y \to X$ such that $G \circ F \geqslant \mathbf{id}_X$ and $F \circ G \geqslant \mathbf{id}_Y$.

The notion of a Galois connection between ordered sets goes back to Évariste Galois' work on the automorphism groups of a field. Birkhoff [2, p. 24], [3, p. 122] defined the concept of *polarity* between power sets; Ore [30, p. 495] introduced Galois connections in the general setting of ordered sets as well as the terms *Galois correspondence* (for each of the mappings) and *Galois connexion* (for the pair of mappings). Grätzer [16, p. 69], Gierz et al. [15, p. 22], Davey and Priestley [8, p. 155], and Blyth [5, p. 14] use the term *Galois connection*.

The preordered sets X and Y in the definition need not be lattices:

Example 4.2. Let $X = \{0, 1\}$ and let Y be any preordered set. Then (F, G), where $F : X \to Y, G : Y \to X$, is a Galois connection if and only if Y possesses a largest element $\mathbf{1}_Y, F(0) = \mathbf{1}_Y$, and $G = \chi_B$, the characteristic function of the down-set $B = \{y \in Y; y \leqslant F(1)\}$. The equivalence class of $F(1)$ determines everything.

It is well known and easy to prove that if (F, G) is a Galois connection, then $G \circ F$ is in fact a cleistomorphism in X; Ore [30, p. 496]. Conversely, any cleistomorphism in X can be obtained in this way for a suitable choice of Y and G (this follows from Gierz et al. [14, p. 23, 3.13], [15, p. 29,O-3.13]; see also [24]; Proposition 2.2). However, as remarked in (2002b:2038), this rather formal result "is, in sense, completely uninteresting"; it is more like a tautology. By contrast, there are many situations in mathematics where $\kappa : X \to X$ can be factorized as $\kappa = G \circ F$ for some $F : X \to Y$ and $G : Y \to X$ which are very interesting and give new insights. Examples include Galois theory, convex analysis (Example 5.13), and image processing (Examples 5.7, 5.11 and 5.12).

Proposition 4.3. *Given preordered sets X, Y, a pair (F, G) of mappings $F : X \to Y$ and $G : Y \to X$ is a Galois connection if and only if* hypo $F = (\text{hypo } G)^{\smile}$.

Birkhoff [3, p. 124] attributes this result to J. Schmidt.

Proof. If (F, G) is a Galois connection and $x \leqslant G(y)$, then $F(x) \geqslant F(G(y))$ since F is decreasing, and since $F(G(y)) \geqslant y$, we get $F(x) \geqslant y$; we have proved that $(\text{hypo } G)^{\smile} \subset \text{hypo } F$. The opposite inclusion follows by symmetry.

Conversely, if hypo $F = (\text{hypo } G)^{\smile}$ and $y \leqslant F(x)$, we deduce that $x \leqslant G(y)$, in particular, taking $y = F(x)$, that $x \leqslant G(F(x))$. Also, if $x' \leqslant x \leqslant G(y)$ we deduce that $y \leqslant F(x)$ and $y \leqslant F(x')$. Since we now know that this holds for $y = F(x)$, we conclude that $F(x) \leqslant F(x')$, hence that F is decreasing. That G is decreasing and that $y \leqslant F(G(y))$ follows by symmetry. □

If we introduce increasing mappings $g = F^{\text{op}} : X \to Y^{\text{op}}$ and $h = G_{\text{op}} : Y^{\text{op}} \to X$, the equality hypo $F = (\text{hypo } G)^{\smile}$ can be written epi $g = (\text{hypo } h)^{\smile}$, which is just (3.6) with equality, taking $h = g_{[-1]}$.

4.2. Residuation

A mapping $g : X \to Y$ of an ordered set X into an ordered set Y is said to be *residuated* if for every $b \in Y$ there is an element $a \in X$ such that the inverse image of $\{y \in Y; y \leqslant b\}$ is equal to $\{x \in X; x \leqslant a\}$ (Blyth and Janowitz [6, p. 11]; Blyth [5, p. 7]). It is equivalent to require that there exists a mapping $h : Y \to X$ such that $(g^{\text{op}}, h_{\text{op}})$ is a Galois connection. The mapping h is proved to be unique and is called the *residual* of g. It is equal to $g_{[-1]}$.

Residuation theory goes back at least to a paper by Ward and Dilworth [44], then in an algebraic setting. Algebraically, residuation is a kind of dual to multiplication. In a lattice L with a multiplication one fixes an element c and assumes that, given y, the set of all x such that $cx \leqslant y$ has a largest element, which is denoted by $y : c$ (we consider for simplicity only the commutative case). We see that this is $g_{[-1]}$ if $g : L \to L$ is the mapping $g(x) = cx$. Thus $cx \leqslant y$ if and only if $x \leqslant y : c$.

4.3. Adjunctions

Gierz et al. [15, p. 22] define a *Galois connection* or *adjunction* as a pair of increasing mappings $(h, k), h : X \to Y, k : Y \to X$ satisfying hypo $h = (\text{epi } k)^{\smile}$. This aspect probably originates in logic.

If (h, k) is an adjunction, Gierz et al. [15, p. 22], Heijmans and Ronse [20, p. 264], and Davey and Priestley [8, p. 156] call h the *upper adjoint* and k the *lower adjoint* in the adjunction. As we shall see, in an adjunction (h, k) we have $k = h^{[-1]}$ and $h = k_{[-1]}$, so each mapping determines the other.

The definition of an adjunction means that $(F, G) = (h^{\text{op}}, k_{\text{op}})$ is a Galois connection in the sense of Definition 4.1. (We shall use the last-mentioned definition in the sequel.) Gierz et al. [15, p. 22] require that h and k be increasing, but this is superfluous as we have seen in Proposition 4.3.

5. Examples

The purpose of this section containing several examples is twofold. The first examples are very simple and just intended to illustrate the concepts and show that some implications—that one could have expected to hold—do not hold. The Examples 5.6, 5.7 and 5.8 and 5.11–5.14 supplement Subsections 4.1, 4.2 and 4.3 in showing that the definitions have a good meaning and yield some insight in a variety of situations.

Example 5.1. Supremum not maximum.

Take $X = \mathscr{P}(E)$, where E is a finite set of cardinality $n \geqslant 2$, and $Y = [0, n]_{\mathbf{Z}} = \{y \in \mathbf{Z}; 0 \leqslant y \leqslant n\}$, two complete lattices, and define $g(A) = \operatorname{card}(A),\ A \in X$. Then $g_{[-1]}(0) = \emptyset = \mathbf{0}_X$ and $g_{[-1]}(y) = E = \mathbf{1}_X$ for all $y \in [1, n]_{\mathbf{Z}}$. Thus $g_{[-1]}(1) = E = \mathbf{1}_X$, and the set $\{A \in X; g(A) \leqslant 1\}$ is not an ideal. It therefore comes as no surprise that, for $n = 2, \operatorname{card}(\text{hypo } g_{[-1]}) = 25 > 20 = \operatorname{card}(\text{epi } g)$ (see Fig. 1).

We also find that $g^{[-1]}(y) = \emptyset = \mathbf{0}_X$ when $y < n$ and $g^{[-1]}(y) = E = \mathbf{1}_X$ when $y = n$.

Example 5.2. Non-residuated mappings.

Take $X = Y = L$, a complete lattice, in Definition 3.1, fix an element c of L, and define a mapping $g : L \to L$ by $g(x) = x \vee c, x \in L$. In this case, the supremum in (3.1) is a maximum if $y \geqslant c$ but only then. Thus g is not residuated unless $c = \mathbf{0}$; also, it is a dilation only if $c = 0$ (indeed, $g(\mathbf{0}) = c$ while $\bigvee_{j\in\emptyset} g(x_j) = \mathbf{0}$). But it is easy to determine its lower inverse: $g_{[-1]}(y) = y$ if $y \geqslant c$ and $g_{[-1]}(y) = \mathbf{0}$ otherwise. We have

$$\text{epi } g = \{(x, y) \in L^2; y \geqslant x \vee c\},$$

while

$$(\text{hypo } g_{[-1]})^{\check{}} = \text{epi } g \cup \{(\mathbf{0}, y) \in L^2; y \ngeqslant c\},$$

so that

$$(\text{hypo } g_{[-1]})^{\check{}} \setminus \text{epi } g = \{(\mathbf{0}, y) \in L^2; y \ngeqslant c\} \neq \emptyset \text{ if } c \neq \mathbf{0}.$$

We have $g(g_{[-1]}(y)) \leqslant y$ if and only if $y \geqslant c$.

For the upper inverse, we can only say that $g^{[-1]}(y) = \mathbf{0}$ if $y \leqslant c$ and that $g^{[-1]}(y) \leqslant y$ for $y \nleqslant c$. Both equality and strict inequality can occur here as we shall see.

Example 5.3. We let g be as in Example 5.2 and assume in addition that L is totally ordered. Then g is an erosion. We have already determined $g_{[-1]}$ in Example 5.2, and we know that $g^{[-1]}(y) = \mathbf{0}$ for $y \leqslant c$. In the case of total order we have $g^{[-1]}(y) = y$ for all $y > c$. In the notation which Singer [40, p. 335] uses for $L = [-\infty, +\infty]$, we can write $g^{[-1]}(y) = y \top c, y \in L$. Thus $g_{[-1]}$ and $g^{[-1]}$ are equal except for $y = c$ if $c \neq \mathbf{0}$; there we get $g^{[-1]}(c) = \mathbf{0} \leqslant c = g_{[-1]}(c)$. Moreover we have equality in (3.7):

$$\text{hypo } g = (\text{epi } g^{[-1]})^{\check{}} = \{(x, y) \in L^2; y \leqslant x \vee c\},$$

which, in view of Theorem 6.8 means that $g^{[-1]}$ is dually residuated with dual residual g; equivalently that $(g^{\text{op}}, (g^{[-1]})_{\text{op}})$ is a Galois connection.

Example 5.4. Let now L be $[0,1]^2$, the Cartesian product of two intervals. The lower inverse is already known from Example 5.2. The upper inverse is

$$g^{[-1]}(y) = \begin{cases} \mathbf{0}, & y \leqslant c; \\ (0, y_2), & y_1 \leqslant c_1,\ y_2 > c_2; \\ (y_1, 0), & y_1 > c_1,\ y_2 \leqslant c_2; \\ y, & y_1 > c_1, y_2 > c_2. \end{cases}$$

Thus strict inequality in $g^{[-1]}(y) \leqslant y$ can occur. We have hypo $g = (\text{epi } g^{[-1]})^{\check{}}$.

Example 5.5. Let now L be $\{0,1\}^2$ with the coordinatewise order, and let g be as in Example 5.2. We choose $c = (1,0)$ and denote $(0,1)$ by a, so that L consists of the four elements $\mathbf{0} = (0,0), a = (0,1), c = (1,0)$, and $\mathbf{1} = (1,1)$. From Example 5.2 we know that $g_{[-1]}(y) = y$ if $y \geqslant c$ and $g_{[-1]}(y) = \mathbf{0}$ otherwise. Thus

$$g_{[-1]}(y) = \begin{cases} \mathbf{0}, & y = \mathbf{0}; \\ \mathbf{0}, & y = a; \\ c, & y = c; \\ \mathbf{1}, & y = \mathbf{1}. \end{cases}$$

We find that

$$(\text{hypo} g_{[-1]})^{\check{}} \setminus \text{epi } g = \{(\mathbf{0}, \mathbf{0}), (\mathbf{0}, a)\} \neq \emptyset.$$

Thus g is not residuated.

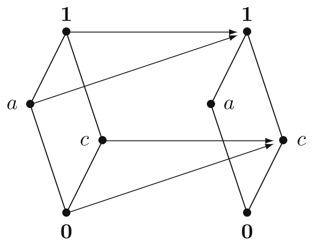

Fig. 2. The mapping g of Example 5.5.

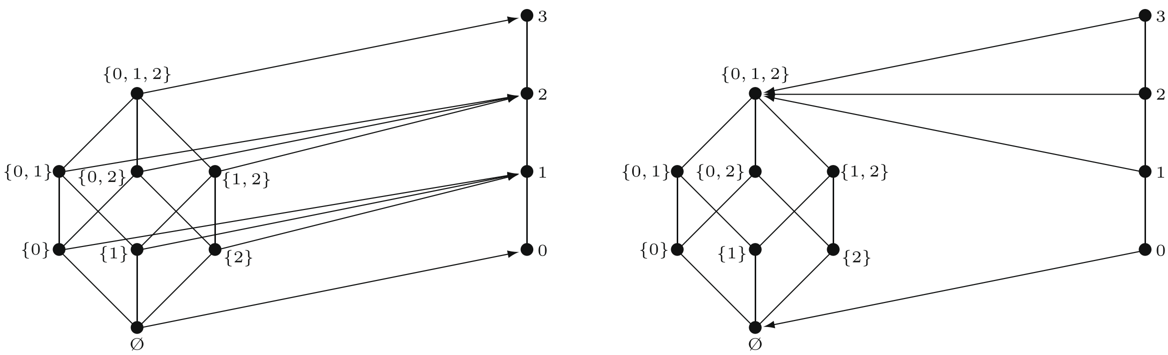

Fig. 1. The mapping g of Example 5.1 (left) and its lower inverse $g_{[-1]}$ in the case $n = 2$. One can see that card(epi g) = 20 and that card(hypo $g_{[-1]}$) = 25.

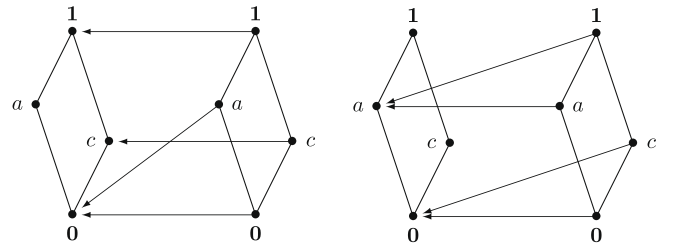

Fig. 3. The lower inverse $g_{[-1]}$ (left) and the upper inverse $g^{[-1]}$ of Example 5.5.

We also find that

$$g^{[-1]}(y) = \begin{cases} \mathbf{0}, & y = \mathbf{0}; \\ a, & y = a; \\ \mathbf{0}, & y = c; \\ a, & y = \mathbf{1}. \end{cases}$$

The infimum is in all cases a minimum, meaning that $g^{[-1]}$ is dually residuated, in other words, $(g^{\mathrm{op}}, (g^{[-1]})_{\mathrm{op}})$ is a Galois connection. We have equality in (3.7):

$$\text{hypo}\, g = (\text{epi}\, g^{[-1]})^{\vee} = L^2 \setminus \{(\mathbf{0}, a), (\mathbf{0}, \mathbf{1}), (c, a), (c, \mathbf{1})\} (\text{See Figs. } 2, 3).$$

Example 5.6. Calculating with infinities.

Let now L be the complete lattice $[-\infty, +\infty]$ of extended real numbers and define

$$S_c(x) = x \dot{+} c, \quad s_c(x) = x \underset{\cdot}{+} c, \quad x \in [-\infty, +\infty],$$

where c is a constant in $[-\infty, +\infty]$. Here $\dot{+}$ and $\underset{\cdot}{+}$ denote *upper* and *lower addition*, defined as extensions of addition $\mathbf{R}^2 \to \mathbf{R}$ to commutative mappings $[-\infty, +\infty]^2 \to [-\infty, +\infty]$ defined by the requirements

$$\begin{aligned} & x \dot{+} (+\infty) = +\infty \quad \text{for all } x \in [-\infty, +\infty]; \\ & x \dot{+} (-\infty) = -\infty \quad \text{for all } x \in [-\infty, +\infty[; \quad \text{and} \\ & x \underset{\cdot}{+} y = -((-x) \dot{+} (-y)) \quad \text{for all } x, y \in [-\infty, +\infty]. \end{aligned} \tag{5.1}$$

Convenient rules in the calculus with these additions are

$$\inf_{j \in J} (c \dot{+} a_j) = c \dot{+} \inf_{j \in J} a_j \quad \text{and} \quad \sup_{j \in J} (c \underset{\cdot}{+} a_j) = c \underset{\cdot}{+} \sup_{j \in J} a_j,$$

which hold for all index sets J and all elements $a_j, c \in [-\infty, +\infty]$. We also note the equivalence

$$a \dot{+} b \geqslant c \Longleftrightarrow a \geqslant c \underset{\cdot}{+} (-b), \quad a, b, c \in [-\infty, +\infty]. \tag{5.2}$$

We note that $S_c = s_c$ for all real numbers c. The mapping s_c is residuated for all $c \in [-\infty, +\infty]$, and $(s_c)_{[-1]} = S_{-c}$. Also $(S_c)^{[-1]} = s_{-c}$ and the infimum is a minimum. But neither $S_{-\infty}$ nor $S_{+\infty}$ is residuated, although we have $(S_{-\infty})_{[-1]} = S_{+\infty}$ and $(S_{+\infty})_{[-1]} = S_{-\infty}$. Similarly, $(s_{-\infty})^{[-1]} = s_{+\infty}$ and $(s_{+\infty})^{[-1]} = s_{-\infty}$.

Example 5.7. Processing binary images.

This field has been my main source of inspiration, and examples abound. A binary image consists of pixels, e.g., spots on a computer screen, and each pixel can be given an address which is a pair of integers. Thus we let $\mathbf{Z}^2$ or $\mathbf{Z}^n$ be the set of addresses. (This does not mean that each pixel must be a square; in fact a hexagonal or triangular tessellation can be given addresses in $\mathbf{Z}^2$ just like the pixels in a rectangular tessellation.) A very common kind of dilation is the operation on $\mathscr{P}(G)$ defined by the Minkowski sum,

$$\delta_S(A) = A + S = \{x + y; x \in A, y \in S\}, \quad A \in \mathscr{P}(G),$$

where S is a fixed subset of G, the structuring element. Here G is any abelian group, for instance $\mathbf{Z}^n$ or $\mathbf{R}^n$, in particular $\mathbf{Z}^2$. In fact, any dilation which commutes with all translations of the group is of this form. Its lower inverse is

$$(\delta_S)_{[-1]}(B) = \bigcup_A (A; \delta_S(A) \subset B) = \{x; x + S \subset B\}, \quad B \in \mathscr{P}(G),$$

and it is well known that it is an erosion (see Figs. 4–6).

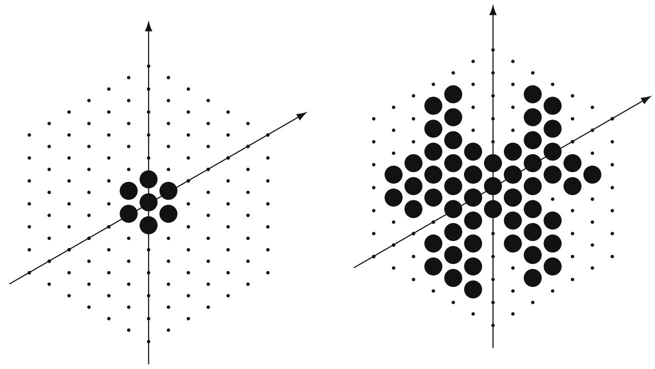

Fig. 4. To the left a structuring element S of Example 5.7 whose seven pixels have addresses $(0,0), (1,0), (0,1), (-1,1), (-1,0), (0,-1), (1,-1)$ in a Cartesian coordinate system—a Cartesian system need not be orthogonal. To the right an image A to be subjected to the dilation $\delta = \delta_S$, the erosion $\delta_{[-1]} = \varepsilon$ as well as the cleistomorphism $\kappa = \varepsilon \circ \delta$ and the anoiktomorphism $\alpha = \delta \circ \varepsilon$.

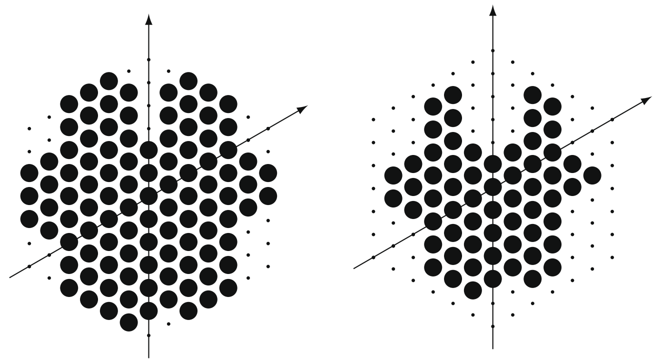

Fig. 5. To the left the dilated set $\delta(A) = \delta_S(A)$, S being as in Fig. 4 to the left and A as in Fig. 4 to the right. To the right the closed set $\kappa(A) = \varepsilon(\delta(A))$. We have $\text{card}(\kappa(A) \setminus A) = 61 - 55 = 6$. We note that the lower, narrow gulf has been filled in, while the upper, wider gulf has been preserved.

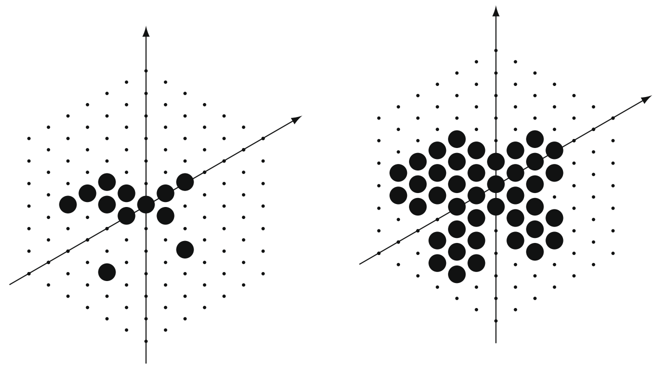

Fig. 6. To the left the eroded set $\varepsilon(A) = (\delta_S)_{[-1]}(A)$, S being as in Fig. 4 to the left and A as in Fig. 4 to the right. To the right the open set $\alpha(A) = \delta(\varepsilon(A))$. We have $\text{card}(A \setminus \varepsilon(A)) = 55 - 41 = 14$. We note that the narrow arms (the two upper arms and the arm to the right) have disappeared, while the thicker arm to the left has survived.

Example 5.8. Infimal and supremal convolution.

Let G be an abelian group—think of $\mathbf{Z}^n$ or $\mathbf{R}^n$. We define for arbitrary functions $f, g \in [-\infty, +\infty]^G$,

$$(f \sqcap g)(x) = \inf_{y \in G}(f(x-y) \dot{+} g(y)), \quad (f \sqcup g)(x) = \sup_{y \in G}(f(x-y) \underset{\cdot}{+} g(y)), \quad x \in G. \tag{5.3}$$

Here $\dot{+}$ and $\underset{\cdot}{+}$ are defined by (5.1). We call the operation $\sqcap$ *infimal convolution* and $\sqcup$ *supremal convolution*. Clearly $f \sqcup g = -((-f) \sqcap (-g))$, so from a theoretical point of view it is enough to study one of them; however, it is convenient to keep both operations. For a survey of the properties and many applications of these convolutions, see Strömberg [43]; for applications see also Heijmans and Ronse [20], Heijmans and Maragos [18], and Heijmans and Molchanov [19].

We note that these convolutions are just Minkowski addition in another dimension:

$$\begin{aligned} &\mathrm{epi}_s^F\ (f \sqcap g) = \mathrm{epi}_s^F\ (f) + \mathrm{epi}_s^F\ (g), \quad f, g \in [-\infty, +\infty]^G, \quad \text{and} \\ &\mathrm{hypo}_s^F(f \sqcup g) = \mathrm{hypo}_s^F(f) + \mathrm{hypo}_s^F\ (g), \quad f, g \in [-\infty, +\infty]^G, \end{aligned} \tag{5.4}$$

where Minkowski addition is now in the abelian group $G \times \mathbf{R}$, and $\mathrm{epi}_s^F\ (f) = \mathrm{epi}_s\ (f) \cap (G \times \mathbf{R})$ is the *finite strict epigraph* of f.

We may study equations $f \sqcap s = g$ with s and g as given functions and f as an unknown function, but in general such equations cannot be solved, and if so, they do not have a unique solution. However, if we pass to inequalities, something useful can be said:

Proposition 5.9. *Let G be an abelian group, fix two functions $s, t \in [-\infty, +\infty]^G$, and define two operations on functions $G \to [-\infty, +\infty]$ by*

$$\delta(f) = f \sqcap s \quad \text{and} \quad \varepsilon(g) = g \sqcup t, \quad f, g \in [-\infty, +\infty]^G.$$

Then $\mathrm{hypo}(\delta) = \mathrm{epi}(\varepsilon)^{\vee}$ *if and only if* $t(y) = -s(-y), y \in G$.

Proof. We find that $\delta(f) \geqslant g$ if and only if $f(x-y) \dot{+} s(y) \geqslant g(x)$ for all $x, y \in G$, and that $f \geqslant \varepsilon(g)$ if and only if $f(x) \geqslant g(x-y) \underset{\cdot}{+} t(y)$ for all $x, y \in G$. We now invoke the equivalence (5.2), which shows that the condition $\delta(f) \geqslant g$ can be written $f(x-y) \dot{+} (-g(x)) \geqslant -s(y)$ for all $x, y \in G$, equivalently

$$f(x) \dot{+} (-g(x-y)) \geqslant -s(-y) \text{ for all } x, y \in G,$$

and that the condition $f \geqslant \varepsilon(g)$ can be written

$$f(x) \dot{+} (-g(x-y)) \geqslant t(y) \text{ for all } x, y \in G.$$

Thus the equivalence holds for all $f, g \in [-\infty, +\infty]^G$ if $t(y) = -s(-y)$ for all $y \in G$ and only then. □

We shall also need a similar result for nonnegative functions:

Proposition 5.10. *Let G be an abelian group, fix two functions $s \in [0, +\infty]^G$ and $t \in [-\infty, 0]^G$, and define two operations on functions $G \to [0, +\infty]$ by*

$$\delta(f) = f \sqcap s \quad \text{and} \quad \varepsilon^+(g) = (g \sqcup t)^+ = (g \sqcup t) \vee 0, \quad f, g \in [0, +\infty]^G.$$

Then $\mathrm{hypo}(\delta) = \mathrm{epi}(\varepsilon^+)^{\vee}$ *if and only if* $t(y) = -s(-y)$ *for all* $y \in G$.

Proof. The proof is quite similar to the previous proof. Note that, for all $f \in [0, +\infty]^G, f \geqslant \varepsilon(g)$ if and only if $f \geqslant \varepsilon^+(g)$. □

The mapping δ is an erosion for the order used. However, as we shall see in the next example, it is natural to use the order defined by the functions e^{-f} rather than by f, which explains that δ will be a dilation and ε^+ an erosion when operating on these functions.

Example 5.11. Processing gray-scale images.

The binary images on a set X form a complete lattice $\mathscr{P}(X)$ isomorphic to $\{0, 1\}^X$. A gray-scale image on X is given by a function $\varphi : X \to P$ on X, where P is a subset of $[0, 1]$ representing gray-levels in the image, for instance

$$P = \left\{0, \frac{1}{255}, \frac{2}{255}, \ldots, \frac{254}{255}, 1\right\}$$

with 2^8 levels. If P is closed, P^X is a complete lattice, actually a sublattice of $[0, 1]^X$.

How to define dilations and erosions on gray-scale images on an abelian group G? A naive way would be to use convolution,

$$(\varphi * \sigma)(x) = \sum_{y \in G} \varphi(x-y)\sigma(y), \quad x \in G,$$

possibly scaled to the interval $[0, 1]$ by dividing by the supremum or taking just $(\varphi * \sigma) \wedge 1$ for any two functions $\varphi, \sigma : G \to [0, 1]$, but these operations do not define dilations. So one could try instead with

$$\delta_\sigma(\varphi) = \sup_{y \in G} \varphi(x-y)\sigma(y), \quad \varphi \in [0, 1]^G,$$

where $\sigma : G \to [0, 1]$ is a given function. This does define a dilation. But if the set P of values is finite, it yields a function with values in P only if σ takes values in $\{0, 1\}$. This definition is one of several as we shall see; in the notation of Bloch and Maître [4, p. 1344] we have $\delta_\sigma = D3_\sigma$.

It will be convenient to formulate these arguments in terms of infimal and supremal convolution.

To any set A we associate its *indicator function* $\mathbf{ind}_A = -\log \chi_A$; the characteristic function is $\chi_A = e^{-\mathbf{ind}_A}$. Similarly, to any gray-scale image φ we associate its *indicator function* $\mathbf{ind}_\varphi = -\log \varphi$.

We now note that the Minkowski sum $A + B$ of two sets A and B satisfies

$$\mathbf{ind}_{A+B} = \mathbf{ind}_A \sqcap \mathbf{ind}_B.$$

This makes it natural to define a dilation δ by a fixed gray-scale image σ and operating on arbitrary gray-scale images φ by the formula

$$\mathbf{ind}_{\delta(\varphi)} = \mathbf{ind}_\varphi \sqcap \mathbf{ind}_\sigma, \quad \mathbf{ind}_\varphi \in [0, +\infty]^G, \tag{5.5}$$

and, as we have seen in Proposition 5.10, its lower inverse $\delta_{[-1]} = \varepsilon^+$ is given by

$$\mathbf{ind}_{\varepsilon^+(\psi)} = (\mathbf{ind}_\psi \sqcup (-\mathbf{ind}_\tau))^+, \quad \mathbf{ind}_\psi \in [0, +\infty]^G, \tag{5.6}$$

where $\tau(x) = \sigma(-x)$. And $\kappa = \varepsilon^+ \circ \delta$ is a cleistomorphism; $\alpha = \delta \circ \varepsilon^+$ an anoiktomorphism (Corollary 6.14).

Example 5.12. Fuzzy logic and fuzzy mathematical morphology.

A subset A of a set E, called a *crisp set* in this context, can be represented by its characteristic function $\chi_A : E \to \{0, 1\}$. A fuzzy subset A of a set E can in the same way be represented by its *membership function* $\mu_A : E \to P$, where the value $\mu_A(x)$ represents the degree (or probability) that x belongs to A. The intersection $A \cap B$ of two sets can be represented, respectively, by

$$\chi_{A \cap B} = \chi_A \wedge \chi_B \quad \text{and} \quad \mu_{A \cap B} = \mu_A \wedge \mu_B;$$

similarly the union is represented by

$$\chi_{A \cup B} = \chi_A \vee \chi_B \quad \text{and} \quad \mu_{A \cup B} = \mu_A \vee \mu_B.$$

Thus the family $\mathscr{P}(E)$ of all crisp sets is isomorphic to the complete lattice $\{0, 1\}^E$, and the family of all fuzzy sets is isomorphic to the lattice $[0, 1]^E$. But as shown by (5.4), the calculus of fuzzy sets can be reduced to calculus of crisp sets at the expense of adding one more dimension.

So far, the fuzzy sets are just like gray-scale images. However, fuzzy logic goes beyond the gray-scale images in generalizing also the minimum and maximum.

In addition to the operations of minimum and maximum there are many other operations on fuzzy sets. We say that a function $c : [0,1]^2 \to [0,1]$ is a *conjunctor* if its restriction to $\{0,1\}^2$ is Boolean conjunction; similarly we say that $d : [0,1]^2 \to [0,1]$ is a *disjunctor* if its restriction to $\{0,1\}^2$ is Boolean disjunction; moreover a *negator* is a function $n : [0,1] \to [0,1]$ such that $n(0) = 1, n(1) = 0$ (Sladoje [41, p. 18–19]). Finally a function $i : [0,1]^2 \to [0,1]$ will be called an *implicator* if $i(0,0) = i(0,1) = i(1,1) = 1$ and $i(1,0) = 0$.

Deng and Heijmans [10, p. 5] use the terms *fuzzy conjunction* or just *conjunction* and *fuzzy implication* or *implication* for a conjunctor which is increasing in both arguments, and an implicator which is decreasing in the first and increasing in the second, respectively.

Examples of conjunctors and disjunctors include

$$\begin{aligned}
&c_M(a,b) = a \wedge b, && d_M(a,b) = a \vee b, && a,b \in [0,1],\\
&c_A(a,b) = ab, && d_A(a,b) = a+b-ab, && a,b \in [0,1], \text{ and}\\
&c_L(a,b) = (a+b-1) \vee 0, && d_L(a+b) \wedge 1, && a,b \in [0,1];
\end{aligned}$$

the second ones are called *algebraic* and the last ones are called the *Łukasiewicz conjunctor* and *disjunctor*.

Given a conjunctor c and a disjunctor d we define the *c-intersection* $A \cap_c B$ and the *d-union* $A \cup_d B$ by their membership functions

$$\mu_{A\cap_c B}(x) = c(\mu_A(x), \mu_B(x)), \quad \mu_{A\cup_d B}(x) = d(\mu_A(x), \mu_B(x)), \quad x \in E.$$

It is customary to require more of the functions c and d. We require $\cap_c$ and $\cup_d$ to be commutative and associative. This is guaranteed if

$$c(a,b) = c(b,a), \quad d(a,b) = d(b,a), \quad a,b \in [0,1], \quad \text{and} \tag{5.7}$$

$$c(c(a,b),z) = c(a,c(b,z)), \; d(d(a,b),z) = d(a,d(b,z)), \; a,b,z \in [0,1]. \tag{5.8}$$

We also require that 1 be a neutral element for c and 0 a neutral element for d:

$$c(a,1) = c(1,a) = a, \quad d(a,0) = d(0,a) = a, \quad a \in [0,1]. \tag{5.9}$$

Conditions (5.7)–(5.9) imply that the fuzzy sets form a commutative semigroup with neutral element under $\cap_c$ as well as under $\cup_d$. Finally, it is natural to require that

$$c \text{ and } d \text{ are increasing in each variable.} \tag{5.10}$$

It follows from (5.7)–(5.10) that both semigroups have a zero element (the empty set Ø for $\cap_c$ and the full set E for $\cup_d$), and that c and d are uniquely determined on the boundary of $[0,1]^2$; in particular at the four corners $\{0,1\}^2$, which implies that $A\cap_c B$ and $A \cup_d B$ agree with $A \cap B$ and $A \cup B$ when A and B are crisp sets.

One says that a function c or d satisfying (5.7)–(5.10) is a *T-norm* or a *T-conorm*, respectively. Clearly the minimum is the largest of all *T*-norms, and the Łukasiewicz *T*-norm c_L is the largest convex *T*-norm.

Translated to the language of fuzzy sets, (5.5) and (5.6) mean that the dilation δ_S by a fuzzy set S with $s = \mathbf{ind}_S$ has lower inverse $(\varepsilon_T)^+$ if and only if $t = -\mathbf{ind}_T$ where $T = \{-s; s \in S\}$.

More general dilations can now be defined as

$$\delta_\sigma(\varphi) = \sup_{y\in G} c(\varphi(x-y), \sigma(y)), \quad \varphi \in [0,1]^G,$$

for any conjunctor c. In their survey article Bloch and Maître [4, p. 1344] denote it by $D6_\sigma$ and explore its properties. The case already considered is when $c(a,b) = c_A(a,b) = ab$; we may instead take $c(a,b) = c_M(a,b) = a \wedge b$ to get the dilation introduced by Kaufmann and Gupta [21] and denoted by $D2_\sigma$ by Bloch and Maître [4, p. 1344]; it can be expressed using a restricted infimal convolution.

Now fix an element $a \in [0,1]$ and define a mapping $g_a(x) = \varphi(a,x)$ for a function φ of two variables. It is easy to see that if φ is a conjunctor, then the lower inverse $(g_a)_{[-1]}$ defines an implicator $i(a,b) = (g_a)_{[-1]}(b)$. Several such implicators have been studied by Deng and Heijmans [10, p. 5] and Deng [9, p. 3]. They are used in fuzzy logic.

If $\varphi = c_M = \min$, the implicator $i_M(a,b) = (g_a)_{[-1]}(b)$ is equal to 1 if $b \geqslant a$ and equal to b when $b < a$ (cf. Example 5.2). The upper inverse satisfies $(g_a)^{[-1]}(b) = 1$ for $b > a$ and $(g_a)^{[-1]}(b) = b$ for $b \leqslant a$ (cf. Example 5.3). Hence the two inverses are equal except for $b = a$.

Taking instead $\varphi = c_A$, we find that $i_A(a,b) = (g_a)_{[-1]}(b) = (g_a)^{[-1]}(b) = (b/a) \wedge 1$ when $a > 0$, $(g_a)_{[-1]}(b) = (g_a)^{[-1]}(b) = 1$ when $a=0$ and $b>0$, while $(g_a)_{[-1]}(b)=1, (g_a)^{[-1]}(b)=0$ when $a=b=0$. Both are discontinuous at the origin.

Finally, taking φ as the Łukasiewicz *T*-norm c_L, we find that the lower and upper inverses agree except when $b = 0$ and $a < 1$: we get $i_L(a,b) = (g_a)_{[-1]}(b) = (g_a)^{[-1]}(b) = (b-a+1) \wedge 1$ for $0 < b \leqslant 1$, while $(g_a)_{[-1]}(0) = 1-a$, $(g_a)^{[-1]}(0) = 0$; the first is continuous, the second discontinuous on $[0,1[\times\{0\}$.

Example 5.13. Convexity theory.

Given a real vector space E and its algebraic dual $E^\star$ (the space of all linear forms on E), the *Fenchel transform* (Fenchel [12]) of a function $\varphi : E \to [-\infty, +\infty]$ is defined as

$$\tilde\varphi(\xi) = \sup_{x\in E}(\xi\cdot x - \varphi(x)), \quad \xi \in E^\star.$$

It satisfies

$$\tilde\varphi \leqslant f \Longleftrightarrow \tilde f \leqslant \varphi \text{ for all } \varphi \in [-\infty,+\infty]^E \text{ and all } f \in [-\infty,+\infty]^{E^\star}.$$

After a change of order on one of the sides it satisfies the second condition in Proposition 4.3, so that we have a Galois connection (cf. also Theorem 6.8 and its Corollary 6.11). It follows that

$$(\inf_{j\in J} \varphi_j)^\sim = \sup_{j\in J} \tilde\varphi_j \text{ for all families} (\varphi_j)_{j\in J},$$

so that we have an anti-erosion or a duality in the sense of Singer; i.e., after a change of the order relation we have a dilation or erosion (cf. condition (E) in Theorem 6.8). Singer [40] studies several other dualities in convexity theory.

Example 5.14. Rådström's definition of smooth functions on arbitrary sets.

Let X be any set and define for a fixed $m \in \mathbf{N} \cup \{\infty\}$ two mappings $F_m : \mathscr{P}(X^{\mathbf{R}}) \to \mathscr{P}(\mathbf{R}^X)$ and $G_m : \mathscr{P}(\mathbf{R}^X) \to \mathscr{P}(X^{\mathbf{R}})$ by

$$\begin{aligned}
&F_m(\Gamma) = \{\varphi; \varphi\circ\gamma \in C^m(\mathbf{R},\mathbf{R}) \text{ for all } \gamma \in \Gamma\}, \quad \Gamma \in \mathscr{P}(X^{\mathbf{R}}), \quad \text{and}\\
&G_m(\Phi) = \{\gamma; \varphi\circ\gamma \in C^m(\mathbf{R},\mathbf{R}) \text{ for all } \varphi \in \Phi\}, \quad \Phi \in \mathscr{P}(\mathbf{R}^X).
\end{aligned}$$

Obviously the pair (F_m, G_m) is a Galois connection. Hans Rådström (1919–1970) defined, in his unpublished work, smooth functions on a set X by fixing a set of smooth curves Γ and then defining a function $\varphi : X \to \mathbf{R}$ as smooth if it belongs to $F_\infty(\Gamma)$. As far as I know, he did not develop a theory. A basic question is whether, for $m = \infty$, we get the usual C^∞ functions if X is a differential manifold and Γ is the set of C^∞ curves; Rådström conjectured that this is so, and Boman [7] proved it. More precisely, Boman proved that, for finite $m \geqslant 1$,

$$C^m(\mathbf{R}^n,\mathbf{R}) \subset F_m(C^\infty(\mathbf{R},\mathbf{R}^n)) \subset C^{m-1,1}(\mathbf{R}^n,\mathbf{R}).$$

Here $C^m(\mathbf{R}^n,\mathbf{R})$ denotes the space of all functions on $\mathbf{R}^n$ with real values whose derivatives of order at most m exist and are continuous, while $C^{m-1,1}(\mathbf{R}^n,\mathbf{R})$ is the subspace of $C^{m-1}(\mathbf{R}^n,\mathbf{R})$ consisting of functions whose derivates of order $m-1$ are all Lipschitz continuous. Boman proved that the first inclusion here is strict; obviously so is the second. But taking the intersection over all finite $m \geqslant 1$ we see that $F_\infty(C^\infty(\mathbf{R},\mathbf{R}^n)) = C^\infty(\mathbf{R}^n,\mathbf{R})$.

Jan Boman has now obtained an explicit description of $F_m(C^m(\mathbf{R},\mathbf{R}^n))$ for all finite $m \geqslant 1$ (personal communication 2008-09-18).

For $m = \infty$, Petermann [31] developed a formalism in the framework of category theory. Finally Michor [29], Kriegl and Nel [27] and Kriegl and Michor [26] developed a theory for global analysis using smooth curves.

6. Properties of inverses

If, given a mapping $g: X \to Y$, we could find a mapping $u: Y \to X$ such that $(\text{epi}\ u)^\smile = \text{hypo}\ g$ we would be content to have a kind of inverse to g. However, usually the best we can do is to study mappings u with $(\text{epi}\ u)^\smile \supset \text{hypo}\ g$ or mappings v with $(\text{epi}\ v)^\smile \subset \text{hypo}\ g$. This we shall do in the following proposition, which shows that the upper and lower inverses are solutions to certain extremal problems. We shall allow mappings that are not defined everywhere.

Proposition 6.1. *Let X be an ordered set, Y a preordered set, and let $g: X \to Y$, $u: D(u) \to X$ and $v: D(v) \to X$ be mappings, where $D(u)$ and $D(v)$, the domains of definition of u and v, are contained in Y and contain $D(g_{[-1]})$. If*

$$(\text{hypo}\ u)^\smile \cap (X \times D(g_{[-1]})) \subset \text{epi}\ g \cap (X \times D(g_{[-1]})) \subset (\text{hypo}\ v)^\smile,$$

then $u|_{D(g_{[-1]})} \leqslant g_{[-1]} \leqslant v|_{D(g_{[-1]})}$ and

$$(\text{hypo}\ u)^\smile \cap (X \times D(g_{[-1]})) \subset \text{epi}\ g \cap (X \times D(g_{[-1]})) \subset (\text{hypo}\ g_{[-1]})^\smile \subset (\text{hypo}\ v)^\smile.$$

Hence $g_{[-1]}$ is the smallest mapping v such that $(\text{hypo}\ v)^\smile$ contains $\text{epi}\ g \cap (X \times D(g^{[-1]}))$.

Similarly, if $D(u)$ and $D(v)$ contain $D(g^{[-1]})$ and

$$(\text{epi}\ u)^\smile \supset \text{hypo}\ g \cap (X \times D(g^{[-1]})) \supset (\text{epi}\ v)^\smile \cap (X \times D(g^{[-1]})),$$

then $u|_{D(g^{[-1]})} \leqslant g^{[-1]} \leqslant v|_{D(g^{[-1]})}$ and

$$(\text{epi}\ u)^\smile \supset (\text{epi}\ g^{[-1]})^\smile \supset (\text{hypo}\ g) \cap (X \times D(g^{[-1]})) \supset (\text{epi}\ v)^\smile \cap (X \times D(g^{[-1]})).$$

Hence $g^{[-1]}$ is the largest mapping u such that $(\text{epi}\ u)^\smile$ contains $\text{hypo}\ g \cap D(g^{[-1]})$.

The proof is straightforward.

Corollary 6.2. *With g, u and v given as in the proposition, assume that* $(\text{hypo}\ u)^\smile = \text{epi}\ g$ *(thus, in particular, u is defined in all of Y). Then $u = g_{[-1]}$. Similarly, if* $(\text{epi}\ v)^\smile = \text{hypo}\ g$, *then $v = g^{[-1]}$. If also Y is an ordered set, then* $(\text{hypo}\ u)^\smile = \text{epi}\ g$ *implies that $u^{[-1]} = g$ in addition to $u = g_{[-1]}$. Similarly,* $(\text{epi}\ v)^\smile = \text{hypo}\ g$ *implies $v_{[-1]} = g$ in addition to $v = g^{[-1]}$.*

The corollary singles out the special case of adjunctions between X and Y among all pairs $(g, g_{[-1]})$ and adjunctions between Y and X among all pairs $(g^{[-1]}, g)$.

An ideal inverse u would satisfy $u \circ g = \mathbf{id}_X, g \circ u = \mathbf{id}_Y$, and the inverse of u would be g. It is therefore natural to compare $g^{[-1]} \circ g$ and $g_{[-1]} \circ g$ with $\mathbf{id}_X$; $g \circ g^{[-1]}$ and $g \circ g_{[-1]}$ with $\mathbf{id}_Y$; and $(g_{[-1]})^{[-1]}$ and $(g^{[-1]})_{[-1]}$ with g. This is what we shall do in the next three subsections.

6.1. Left inverses

We shall now investigate to what extent $g^{[-1]}$ and $g_{[-1]}$ can serve as left inverses to g.

Proposition 6.3. *Suppose that X is an ordered set and Y a preordered set. Then for all mappings $g: X \to Y$ one has $\mathbf{id}_X|_{X_g} \leqslant g_{[-1]} \circ g$ and $g^{[-1]} \circ g \leqslant \mathbf{id}_X|_{X^g}$, where X_g is the set of all $x \in X$ such that $g(x) \in D(g_{[-1]})$, and X^g is the set of all $x \in X$ such that $g(x) \in D(g^{[-1]})$. The following three conditions are equivalent:*

(α) g is coincreasing;
(β) $g_{[-1]} \circ g = \mathbf{id}_X|_{X_g}$;
(γ) $g^{[-1]} \circ g = \mathbf{id}_X|_{X^g}$.

Proof. If $g(a)$ belongs to $D(g_{[-1]})$, it is clear that $g_{[-1]}(g(a)) = \bigvee_x(x; g(x) \leqslant g(a)) \geqslant a$.

If g is coincreasing, then $\{x; g(x) \leqslant g(a)\}$ is contained in $\{x; x \leqslant a\}$, which, if $x \in X_g$, implies that $g_{[-1]}(g(a)) \leqslant \bigvee_x(x; x \leqslant a) = a$. Thus ($\alpha$) implies ($\beta$).

Conversely, if $g_{[-1]}(g(a)) \leqslant a$, then for all x, $g(x) \leqslant g(a)$ implies $x \leqslant a$. If this is true for all a, then g is coincreasing. So (β) implies (α).

The result on the upper inverse follows by duality. □

Corollary 6.4. *Let X be an ordered set and Y a preordered set. Assume that $g_{[-1]}$ and $g^{[-1]}$ are defined in all of Y. Then $g^{[-1]}(y) \leqslant g_{[-1]}(y)$ for all $y \in \text{im}\,g$, and also for all y majorizing or minorizing $\text{im}\,g$. In particular, $g^{[-1]} \leqslant g_{[-1]}$ if g is surjective.*

Proof. The statement for $y \in \text{im}\,g$ follows directly from the proposition. If y majorizes all elements in $\text{im}\,g$, then $g_{[-1]}(y) = \mathbf{1}_X$, and if y minorizes all elements in $\text{im}\,g$, then $g^{[-1]}(y) = \mathbf{0}_L$. □

Proposition 6.5. *Let X be an ordered set and Y a preordered set. If $g_{[-1]}$ is defined everywhere and $v: Y \to X$ is an increasing mapping such that $\mathbf{id}_X \leqslant v \circ g$, then $g_{[-1]} \leqslant v$. Similarly, if $g^{[-1]}$ is defined in all of Y and $u: Y \to X$ is an increasing mapping such that $u \circ g \leqslant \mathbf{id}_X$, then $u \leqslant g^{[-1]}$. Hence, in view of Proposition 6.3, $g_{[-1]}$ is the smallest increasing mapping v such that $\mathbf{id}_X \leqslant v \circ g$, and $g^{[-1]}$ is the largest increasing mapping u such that $u \circ g \leqslant \mathbf{id}_X$.*

Proof. If u and v are increasing and $u \circ g \leqslant \mathbf{id}_X \leqslant v \circ g$, then $\text{epi}\ g \subset (\text{hypo}\ v)^\smile$ and $(\text{epi}\ u)^\smile \supset \text{hypo}\ g$. We can now apply Proposition 6.1. □

Theorem 6.6. *Let L be a complete lattice and Y a preordered set. Then the following six conditions on a mapping $g: L \to Y$ are equivalent.*

(a) g is coincreasing;
(b) $g^{[-1]} \circ g \geqslant \mathbf{id}_L$;
(c) $g^{[-1]} \circ g = \mathbf{id}_L$;
(d) $g_{[-1]} \circ g \leqslant \mathbf{id}_L$;
(e) $g_{[-1]} \circ g = \mathbf{id}_L$;
(f) $g_{[-1]} \leqslant g^{[-1]}$.

Proof. (a) *implies* (c) *and* (e). If g is coincreasing, we know already from Proposition 6.3 that (c) and (e) hold.

(c) *implies* (b); (e) *implies* (d). This is trivial.
(b) *or* (d) *implies* (a). If (b) or (d) holds, then, in view of Proposition 6.3, they both hold with equality, and g is coincreasing.
(a) *implies* (f). Assume that g is coincreasing and fix an element $y \in Y$. Let $x, x' \in L$ be such that $g(x) \leqslant y \leqslant g(x')$. Then $x \leqslant x'$. Letting x vary, we see that $g_{[-1]}(y) \leqslant x'$. Letting now x' vary, we see that $g_{[-1]}(y) \leqslant g^{[-1]}(y)$. Thus (f) holds.
(f) *implies* (a). If x and x' are given with $g(x) \leqslant g(x')$ we define $y = g(x)$. Then $x \leqslant g_{[-1]}(y)$ and $g^{[-1]}(y) \leqslant x'$. If we know that $g_{[-1]}(y) \leqslant g^{[-1]}(y)$, it follows that $x \leqslant x'$, proving that g is coincreasing. □

6.2. Right inverses

Next we compose $g_{[-1]}$ with g in the other order: we shall see to what extent the inverses we have constructed can serve as right inverses. This will lead to a characterization of dilations—and, by duality, of erosions, anti-erosions and anti-dilations.

Theorem 6.7. *If X is an ordered set, Y a preordered set and $g : X \to Y$ a mapping such that $g_{[-1]}$ is defined everywhere, then the following four properties are equivalent.*

(A) $(\text{hypo}(g_{[-1]}))^{\check{}} \subset \text{epi } g$;
(B) $(\text{hypo}(g_{[-1]}))^{\check{}} = \text{epi } g$;
(C) g is increasing and $(\text{graph}(g_{[-1]}))^{\check{}} \subset \text{epi } g$;
(D) g is increasing and $g \circ g_{[-1]} \leqslant \mathbf{id}_Y$.

Proof. (A) *is equivalent to* (B). This is clear since epi g is always a subset of $(\text{hypo } g_{[-1]})^{\check{}}$; cf. (3.6).

(A) *implies* (C). We note first that g is increasing if (A) holds. Indeed, if $x \leqslant x'$ and we define $y' = g(x')$, then $g_{[-1]}(y') = g_{[-1]}(g(x')) \geqslant x' \geqslant x$ (see Proposition 6.3), which by (A) implies that $g(x) \leqslant y' = g(x')$. (This is a point where we need $g_{[-1]}$ to be defined everywhere.)

That (A) implies $(\text{graph}(g_{[-1]}))^{\check{}} \subset \text{epi } g$ is a consequence of the inclusion $\text{graph}(g_{[-1]}) \subset \text{hypo}(g_{[-1]})$.

(C) *implies* (A). If $x \leqslant g_{[-1]}(y)$ we define $x' = g_{[-1]}(y)$ and note that $x \leqslant x'$ and that $(y, x') \in \text{graph } g_{[-1]}$. If (C) holds, we conclude that $g(x') \leqslant y$. Hence, if g is increasing, $g(x) \leqslant g(x') \leqslant y$, proving (A).

(C) *and* (D) *are equivalent.* (D) is just a rephrasing of (C). □

We now add a new property to the list.

Theorem 6.8. *If L and M are complete lattices and $g : L \to M$ is any mapping, then the following five properties are equivalent.*

(A) $(\text{hypo}(g_{[-1]}))^{\check{}} \subset \text{epi } g$;
(B) $(\text{hypo}(g_{[-1]}))^{\check{}} = \text{epi } g$;
(C) g is increasing and $(\text{graph}(g_{[-1]}))^{\check{}} \subset \text{epi } g$;
(D) g is increasing and $g \circ g_{[-1]} \leqslant \mathbf{id}_M$;
(E) g is a dilation.

By duality we get a similar characterization of erosions.

Singer [40, p. 178, proposition 5.3] proves that (E) and (D) are equivalent (expressed for dualities, i.e., anti-erosions).

Proof. (E) *implies* (A). Suppose that (E) holds. Then if $(y, x) \in \text{hypo } g_{[-1]}$, in other words, if $x \leqslant g_{[-1]}(y)$, we obtain, since g is increasing by hypothesis,

$$g(x) \leqslant g(g_{[-1]}(y)) = g\Big(\bigvee(x; g(x) \leqslant y)\Big) = \bigvee(g(x); g(x) \leqslant y) \leqslant y,$$

which means that $(x, y) \in \text{epi } g$. Thus (A) holds.

(A) *implies* (E). We have already seen that g is increasing if (A) holds (see the proof that (A) implies (C) above). Let now any family $(x_j)_j$ of elements of L be given and define $z = \bigvee g(x_j)$, $w = g(\bigvee x_j)$. Since g is increasing we always have $z \leqslant w$. Is it true that $w \leqslant z$? We note that, by Proposition 6.3, $x_j \leqslant g_{[-1]}(g(x_j)) \leqslant g_{[-1]}(z)$. Taking the supremum over all j we obtain $\bigvee x_j \leqslant g_{[-1]}(z)$, which by (A) implies that $w = g(\bigvee x_j) \leqslant z$. We have proved (E). □

Remark 6.9. We may use (A), (B), (C) or (D) to define dilations $L \to Y$ when Y is only a preordered set.

Corollary 6.10. *If L and M are complete lattices and $g : L \to M$ and $u : M \to L$ are two mappings such that* epi $g = (\text{hypo } u)^{\check{}}$, *then g is a dilation and u is an erosion, and $g_{[-1]} = u$, $u^{[-1]} = g$; $(g^{\text{op}}, u_{\text{op}})$ and $(u_{\text{op}}, g^{\text{op}})$ are Galois connections.*

Proof. It follows from epi $g = (\text{hypo } u)^{\check{}}$ that $g_{[-1]} = u$, hence that (B) in the theorem holds. Since (B) is equivalent to (E), we see that g is a dilation. The rest follows by duality. □

We can sum up the discussion by saying that the study of Galois connections in complete lattices is equivalent to the study of dilations or anti-dilations:

Corollary 6.11. *Let L and M be complete lattices and $F : L \to M$ a mapping. Then the following conditions are equivalent.*

(A) There exists a mapping $G : M \to L$ such that (F, G) is a Galois connection;
(B) The mapping $g = F^{\text{op}}$ is increasing and $g \circ g_{[-1]} \leqslant \mathbf{id}_Y$;
(C) F is an anti-dilation.

Proof. The corollary follows from the theorem and Proposition 4.3. □

6.3. Inverses of inverses

Theorem 6.12. *If L and M are complete lattices and $g : L \to M$ is any mapping, then quite generally $(g_{[-1]})^{[-1]} \leqslant g \leqslant (g^{[-1]})_{[-1]}$. Equality holds at the first place if and only if g is a dilation; at the second place if and only if g is an erosion.*

Proof. By (3.6) we always have epi $g \subset (\text{hypo}\, g_{[-1]})^{\check{}}$, i.e., $y \geqslant g(a)$ implies $g_{[-1]}(y) \geqslant a$. This yields

$$(g_{[-1]})^{[-1]}(a) = \bigwedge(y; g_{[-1]}(y) \geqslant a) \leqslant \bigwedge(y; y \geqslant g(a)) = g(a).$$

If g is a dilation, then, as Theorem 6.8 shows, epi $g = (\text{hypo } g_{[-1]})^{\check{}}$ and equality follows. Conversely, let us note that, in view of (3.6) and (3.7) we always have

$$\text{epi } g \subset (\text{hypo } g_{[-1]})^{\check{}} \subset \text{epi}((g_{[-1]})^{[-1]}).$$

Now if $(g_{[-1]})^{[-1]} = g$, then these inclusions are equalities, and we conclude that epi $g = (\text{hypo}\, g_{[-1]})^{\check{}}$, which according to Theorem 6.8 means that g is a dilation. The last statement follows by duality. □

Theorem 6.13. *If L and M are complete lattices and $\delta : L \to M$ is a dilation, then the lower inverse $\delta_{[-1]} : M \to L$ is an erosion (the residual of δ, or the upper adjoint in the adjunction $(\delta_{[-1]}, \delta)$). Similarly, if $\varepsilon : L \to M$ is an erosion, then the upper inverse $\varepsilon^{[-1]}$ is a dilation.*

Proof. We know that $u = \delta_{[-1]}$ is increasing, so we have, for any family $(y_j)_{j \in J}$, $u(\bigwedge y_j) \leqslant u(y_k)$ for all k; hence $u(\bigwedge y_j) \leqslant \bigwedge u(y_k) = x$. We need to prove the opposite inequality, $x \leqslant u(y)$, where $y = \bigwedge y_j$. From (D) in Theorem 6.8 we learn that $\delta(u(y_k)) \leqslant y_k$ for all k, which implies that $\delta(x) = \delta(\bigwedge u(y_j)) \leqslant y_k$. By (3.6), this implies that $x \leqslant u(y)$. □

Corollary 6.14. *For any dilation $\delta : L \to M$ we have $\delta \circ \delta_{[-1]} \circ \delta = \delta$ and $\delta_{[-1]} \circ \delta \circ \delta_{[-1]} = \delta_{[-1]}$. In particular, $\delta_{[-1]} \circ \delta$ and $\delta \circ \delta_{[-1]}$ are idempotent and therefore ethmomorphisms. The first is a cleistomorphism in L, the second an anoiktomorphism in M. Dually $\varepsilon \circ \varepsilon^{[-1]} \circ \varepsilon = \varepsilon$ and $\varepsilon^{[-1]} \circ \varepsilon \circ \varepsilon^{[-1]} = \varepsilon^{[-1]}$ for any erosion $\varepsilon : L \to M$. Also $\varepsilon^{[-1]} \circ \varepsilon$ and $\varepsilon \circ \varepsilon^{[-1]}$ are idempotent; the first an anoiktomorphism, the second a cleistomorphism.*

Proof. This result is well known. We always have $g_{[-1]} \circ g \geqslant \mathbf{id}_L$ (Proposition 6.3); it follows that $g_{[-1]} \circ g \circ g_{[-1]} \geqslant g_{[-1]}$. If g is increasing, we also get $g \circ g_{[-1]} \circ g \geqslant g$. For dilations we have $\delta \circ \delta_{[-1]} \leqslant \mathbf{id}_M$ (Theorem 6.8), from which we deduce that $\delta \circ \delta_{[-1]} \circ \delta \leqslant \delta$ and $\delta_{[-1]} \circ \delta \circ \delta_{[-1]} \leqslant \delta_{[-1]}$. This shows what we want for dilations; the rest follows by duality. □

7. Division of mappings

We shall now generalize the definitions of upper and lower inverses.

Definition 7.1. Let a set X, a complete lattice M, and a preordered set Y, as well as two mappings $f : X \to M$ and $g : X \to Y$ be given. We define two mappings $f/_{\star}g, f/^{\star}g : Y \to M$ by

$$(f/_{\star}g)(y) = \bigvee_{x\in X}(f(x); g(x) \leqslant_Y y), \quad y \in Y. \tag{7.1}$$

$$(f/^{\star}g)(y) = \bigwedge_{x\in X}(f(x); g(x) \geqslant_Y y), \quad y \in Y. \tag{7.2}$$

We shall call them the *lower quotient* and the *upper quotient* of f and g.

We shall often assume that X, M and Y are all complete lattices, but this is not necessary for the definitions to make sense.

We note that $f/_{\star}g = (f^{\mathrm{op}}/^{\star}g^{\mathrm{op}})^{\mathrm{op}}_{\mathrm{op}}$.

We may also consider $(f/_{\star}g^{\mathrm{op}})_{\mathrm{op}} = (f^{\mathrm{op}}/^{\star}g)^{\mathrm{op}}$ and $(f/^{\star}g^{\mathrm{op}})_{\mathrm{op}} = (f^{\mathrm{op}}/_{\star}g)^{\mathrm{op}}$; explicitly,

$$((f/_{\star}g^{\mathrm{op}}))_{\mathrm{op}}(y) = ((f^{\mathrm{op}}/^{\star}g))^{\mathrm{op}}(y) = \bigvee_{x\in X}(f(x); g(x) \geqslant_Y y), \quad y \in Y; \tag{7.3}$$

$$((f/^{\star}g^{\mathrm{op}}))_{\mathrm{op}}(y) = ((f^{\mathrm{op}}/_{\star}g))^{\mathrm{op}}(y) = \bigwedge_{x\in X}(f(x); g(x) \leqslant_Y y), \quad y \in Y. \tag{7.4}$$

We refrain from generalizing these definitions to the case when the supremum or infimum does not exist for all choices of y.

The quotients $f/_{\star}g$ and $f/^{\star}g$ increase when f increases and they decrease when g increases—just as with division of positive numbers:

If $f_1 \leqslant_M f_2$ and $g_1 \geqslant_Y g_2$, then $f_1/_{\star}g_1 \leqslant_M f_2/_{\star}g_2$ and $f_1/^{\star}g_1 \leqslant_M f_2/^{\star}g_2$.

The mappings $f/_{\star}g$ and $f/^{\star}g$ are always increasing. If $g(x) \leqslant_Y y$, then $f(x) \leqslant_M (f/_{\star}g)(y)$; if $g(x) \geqslant_Y y$, then $f(x) \geqslant_M (f/^{\star}g)(y)$. In particular, if $g(x) = y$, then $(f/^{\star}g)(y) \leqslant_M f(x) \leqslant_M (f/_{\star}g)(y)$.

If we specialize the definitions to the situation when $X = M$ and $f = \mathbf{id}_X$, then $f/_{\star}g = \mathbf{id}_X/_{\star}g = g_{[-1]}$ and $f/^{\star}g = \mathbf{id}_X/^{\star}g = g^{[-1]}$; cf. Definition 3.1.

We note another special case:

Proposition 7.2. *For all mappings $f : X \to M$ we have*

$$f/_{\star}f \leqslant \mathbf{id}_M \leqslant f/^{\star}f$$

and

$$(f/_{\star}f) \circ f = f = (f/^{\star}f) \circ f. \tag{7.5}$$

Proof. These results follow on taking $Y = M$ and $g = f$ in the definition. □

Proposition 7.3. *Let X be an arbitrary subset of a complete lattice M, let $Y = M$, and let g be the inclusion mapping $X \to M$. Then $f/^{\star}g = f^{\diamondsuit}$ and $f/_{\star}g = f_{\diamondsuit}$, where $f^{\diamondsuit}$ is the largest increasing mapping $h : M \to M$ such that $h|_X$ minorizes f, i.e.,*

$$f^{\diamondsuit}(y) = \sup_h(h(y); h \text{ is increasing and } h(x) \leqslant f(x) \quad \text{for all } x \in X);$$

and $f_{\diamondsuit}$ is the smallest increasing mapping k such that $k|_X$ majorizes f, i.e.,

$$f_{\diamondsuit}(y) = \inf_k(k(y); k \text{ is increasing and } k(x) \geqslant f(x) \text{ for all } x \in X).$$

If f itself is increasing, they are in fact extensions of f.

The definitions of $f^{\diamondsuit}$ and $f_{\diamondsuit}$ are taken from Matheron [28, p. 187] and are generalized here to any complete lattice. (Matheron considered the power set of a set and assumed f itself to be increasing.)

If we specialize further, letting also f be the inclusion mapping $X \to M$, we obtain

$$(f/_{\star}f)(y) = f_{\diamondsuit}(y) = \bigvee_{x\in X}(x; x \leqslant y) = y^{\circ} \in M, \tag{7.6}$$

where the last equality defines y°. It is easy to verify that $y \mapsto y^{\circ}$ is an anoiktomorphism. A well-known situation is described in the following example.

Example 7.4. Convexity theory again.

Let M be the complete lattice $[-\infty, +\infty]^E$ of functions on a vector space E with values in the extended reals, let F be a vector subspace of its algebraic dual $E^{\star}$, and let X be the set of all affine functions with linear part in F, i.e., functions of the form $\varphi(x) = \xi(x) + c$ for some linear form $\xi \in F$ and some real constant c. A function f such that $f^{\circ} = f$, where f° is defined by (7.6), is called *X-convex* by Singer [40].

We see that a function on E is X-convex in the sense of Singer if and only if it is equal to the supremum of all its affine minorants belonging to X.

We may ask for a characterization of the X-convex functions. A generalization of Fenchel's theorem to this setting gives the answer: this happens if and only if the function possesses three properties: (a) it is convex in the usual sense; (b) it is lower semicontinuous for the topology $\sigma(E, F)$ on E generated by the linear forms in F; and (c) it does not take the value $-\infty$ except when it is equal to the constant $-\infty$.

Proposition 7.5. *If $f : X \to M$ is increasing and $g : X \to Y$ is coincreasing, then $f/_{\star}g \leqslant f/^{\star}g$.*

Proof. We have

$$(f/_{\star}g)(y) = \bigvee_x(f(x); g(x) \leqslant y) \quad \text{and} \quad (f/^{\star}g)(y) = \bigwedge_{x'}(f(x'); g(x') \leqslant y).$$

If $g(x) \leqslant y \leqslant g(x')$, then $x \leqslant x'$ and $f(x) \leqslant f(x')$. Now take the supremum over all x and the infimum over all x'. □

The quotients are the optimal solutions to an inequality:

Proposition 7.6. *For all mappings $f : X \to M$ and $g : X \to Y$ we have*

$$(f/^{\star}g) \circ g \leqslant f \leqslant (f/_{\star}g) \circ g \tag{7.7}$$

with equality if f is increasing and g is coincreasing. From this we deduce that $(f/^{\star}g)(y) \leqslant (f/_{\star}g)(y)$ for all $y \in \operatorname{im} g$ as well as for all majorants and minorants of $\operatorname{im} g$. In particular, $f/^{\star}g \leqslant f/_{\star}g$ if g is surjective.

Conversely, if $u, v : Y \to M$ are two increasing mappings such that $u \circ g \leqslant f \leqslant v \circ g$, then $u \leqslant f/^{\star}g$ and $v \geqslant f/_{\star}g$. Thus $f/^{\star}g$ is the largest increasing mapping u such that $u \circ g \leqslant f$, and $f/_{\star}g$ is the smallest increasing mapping v such that $f \leqslant v \circ g$.

In the special case $X = Y$ and $g = \mathbf{id}_X$ we obtain

$$f/^{\star}\mathbf{id}_X \leqslant f \leqslant f/_{\star}\mathbf{id}_X,$$

where $f/^{\star}\mathbf{id}_X = f^{\diamondsuit}$ is the largest increasing minorant of f and $f_{\star}/\mathbf{id}_X = f^{\diamondsuit}$ is the smallest increasing majorant of f; when f itself is increasing we therefore get equality.

Proof. The proof is straightforward. The equality in (7.7) follows from Proposition 7.5. □

We next compare the quotient $f/_{\star}g$ and the composition $f \circ g_{[-1]}$ (think of $x/y = x \cdot y^{-1}$ for positive numbers):

Proposition 7.7. *Let $f: X \to M$ be an increasing mapping, assuming X to be an ordered set, and $g: X \to Y$ a mapping such that $g_{[-1]}$ is defined in all of Y. We then have $f/_{\star}g \leqslant f \circ g_{[-1]}$. If we in addition assume that X is a complete lattice and f is a dilation, we have equality here. Similarly, if $g^{[-1]}$ is defined everywhere, $f/^{\star}g \geqslant f \circ g^{[-1]}$, with equality if f is an erosion, assuming X to be a complete lattice. If g is coincreasing, then $f/_{\star}g \leqslant f \circ g_{[-1]} \leqslant f \circ g^{[-1]} \leqslant f/^{\star}g$.*

Proof. The proof is straightforward; see Theorem 6.6 for the last statement. □

Proposition 7.8. *If P is a preordered set and $h: M \to P$ is increasing, we have $h \circ (f/_{\star}g) \geqslant (h \circ f)/_{\star}g$ with equality if h is a dilation. Similarly, $h \circ (f/^{\star}g) \leqslant (h \circ f)/^{\star}g$ with equality if h is an erosion. A special case is Proposition 7.7 (take $X = M$ and $f = \mathbf{id}_X$).*

Proof. The proof is straightforward. □

8. Pullbacks and pushforwards

We shall now see how the notions introduced fit into the study of a very fundamental situation, that of a mapping $f: X \to Y$ pulling back mappings $\psi: Y \to L$ and pushing forward mappings $\varphi: X \to L$.

Definition 8.1. Let three sets X, Y and L be given, as well as a mapping $f: X \to Y$. We define the *pullback* of f, denoted by $f^{\leftarrow}: L^Y \to L^X$ and having as values mappings $f^{\leftarrow}(\psi): X \to L$, by

$$f^{\leftarrow}(\psi) = \psi \circ f \in L^X, \quad \psi \in L^Y. \tag{8.1}$$

If L is a complete lattice, we define the *lower* and *upper pushforwards* denoted by $f_{\rightarrow}, f^{\rightarrow}: L^X \to L^Y$ and yielding as values mappings $f_{\rightarrow}(\varphi), f^{\rightarrow}(\varphi): Y \to L$, by

$$f_{\rightarrow}(\varphi)(y) = \bigvee_{x \in X}(\varphi(x); f(x) = y), \quad y \in Y, \quad \varphi \in L^X, \tag{8.2}$$

and

$$f^{\rightarrow}(\varphi)(y) = \bigwedge_{x \in X}(\varphi(x); f(x) = y), \quad y \in Y, \quad \varphi \in L^X. \tag{8.3}$$

It is customary to write f^{*} for $f^{\leftarrow}$.

If $y \notin \operatorname{im} f$, we obtain $f_{\rightarrow}(\varphi)(y) = \mathbf{0}_L$; the supremum is not attained. Similarly, $f^{\rightarrow}(\varphi)(y) = \mathbf{1}_L$ for these y. Of course there are many other cases when the supremum and infimum in (8.2), (8.3) are not attained. In fact, $f_{\rightarrow}(\varphi)(y)$ and $f^{\rightarrow}(\varphi)(y)$ determine the smallest interval in L which contains the range of φ on the set $\{x; f(x) = y\}$, but the range itself need not be an interval.

If ψ is constant, then $f^{\leftarrow}(\psi)$ is also a constant function, taking the same value.

We note that the lower pushforward defined by (8.2) is actually a lower quotient, $f_{\rightarrow}(\varphi) = \varphi/_{\star}f$, and similarly $f^{\rightarrow}(\varphi) = \varphi/^{\star}f$, namely if we provide Y with the discrete order. The results on quotients are therefore available in this setting.

Proposition 8.2. *Let $f: X \to Y$ and a complete lattice L be given. Then the pullback $f^{\leftarrow}: L^Y \to L^X$ is both a dilation and an erosion.*

Proof. We have to prove that $\bigvee f^{\leftarrow}(\psi_j) = f^{\leftarrow}(\bigvee \psi_j)$, equivalently that $\bigvee(\psi_j \circ f) = (\bigvee \psi_j) \circ f$, which is true by the definition of the supremum in L^X and L^Y. Similarly for the infimum. □

Proposition 8.3. *If X, Y are sets, $f: X \to Y$ a mapping, and L a complete lattice, then*

$$\operatorname{epi} f_{\rightarrow} = (\operatorname{hypo} f^{\leftarrow})^{\check{}} \subset L^X \times L^Y \tag{8.4}$$

and

$$\operatorname{hypo} f^{\rightarrow} = (\operatorname{epi} f^{\leftarrow})^{\check{}} \subset L^X \times L^Y. \tag{8.5}$$

Proof. The statements in the proposition mean that, for all $(\varphi, \psi) \in L^X \times L^Y$, we have

$\psi \geqslant f_{\rightarrow}(\varphi)$ if and only if $\psi \circ f \geqslant \varphi$

and

$\psi \leqslant f^{\rightarrow}(\varphi)$ if and only if $\psi \circ f \leqslant \varphi$.

These statements are easy to prove. □

Corollary 8.4. *Let a mapping $f: X \to Y$ and a complete lattice L be given. Then the lower inverse of the lower pushforward is equal to the pullback, $(f_{\rightarrow})_{[-1]} = f^{\leftarrow}$, and the upper inverse of the pullback is equal to the lower pushforward, $(f^{\leftarrow})^{[-1]} = f_{\rightarrow}$. Also the lower inverse of the pullback is equal to the upper pushforward, $(f^{\leftarrow})_{[-1]} = f^{\rightarrow}$, and the upper inverse of the upper pushforward is equal to the pullback, $(f^{\rightarrow})^{[-1]} = f^{\leftarrow}$. In all cases, the supremum and infimum defining the lower and upper inverses are attainded.*

Proof. The first statement follows from Corollary 6.10 applied to (7.4); the second from Corollary 6.10 applied to (7.5). The statement about the suprema and infima being attained is easy to check. Consider for example

$$(f_{\rightarrow})_{[-1]}(\psi) = \bigvee_{\varphi}(\varphi; f_{\rightarrow}(\varphi) \leqslant \psi) = \bigvee_{\varphi}(\varphi; \varphi \leqslant f^{\leftarrow}(\psi)) = f^{\leftarrow}(\psi),$$

where it is obvious that the supremum is attained. □

Corollary 8.5. *The lower pushforward mapping $f_{\rightarrow}$ is a dilation, and the upper pushforward $f^{\rightarrow}$ is an erosion.*

Proof. By Theorem 6.13, the upper inverse of an erosion is a dilation; hence $f_{\rightarrow} = (f^{\leftarrow})^{[-1]}$ is a dilation. The lower inverse of a dilation is an erosion; hence $f^{\rightarrow} = (f^{\leftarrow})_{[-1]}$ is an erosion. □

Summing up, we have two pairs of mutually inverse mappings: $(f_{\rightarrow})_{[-1]} = f^{\leftarrow}$, both a dilation and an erosion, $(f^{\leftarrow})^{[-1]} = f_{\rightarrow}$, a dilation, and $(f^{\leftarrow})_{[-1]} = f^{\rightarrow}$, an erosion, $(f^{\rightarrow})^{[-1]} = f^{\leftarrow}$, both a dilation and an erosion.

The other inverses $(f^{\rightarrow})_{[-1]}$ and $(f_{\rightarrow})^{[-1]}$ are not so easy to characterize. Let us note the following.

Proposition 8.6. *If there is an element $y \in Y \setminus \operatorname{im} f$ such that $\psi(y) < \mathbf{1}_L$, then $(f^{\rightarrow})_{[-1]}(\psi)(x) = \mathbf{0}_L$ for all x. If on the other hand ψ is identically equal to $\mathbf{1}_L$ in the complement of $\operatorname{im} f$ and x the only point which is mapped to $f(x)$, then, $(f^{\rightarrow})_{[-1]}(\psi)(x) = f^{\leftarrow}(\psi)(x) = \psi(f(x))$, and if x is such that more than one point is mapped to $f(x)$, then $(f^{\rightarrow})_{[-1]}(\psi)(x) = \mathbf{1}_L$. Similarly, if there is a $y \in Y \setminus \operatorname{im} f$ such that $\psi(y) > \mathbf{0}_L$, then $(f_{\rightarrow})^{[-1]}(\psi)(x) = \mathbf{1}_L$ for all x. If on the other hand ψ is identically equal to $\mathbf{0}_L$ in the complement of $\operatorname{im} f$ and x the only point which is mapped to $f(x)$, then, $(f_{\rightarrow})^{[-1]}(\psi)(x) = f^{\leftarrow}(\psi)(x) = \psi(f(x))$, and if x is such that more than one point is mapped to $f(x)$, then $(f_{\rightarrow})^{[-1]}(\psi)(x) = \mathbf{0}_L$.*

We consider now the special case when $L = \{0, 1\}$ and ψ is a characteristic function, $\psi = \chi_B \in \{0, 1\}^Y$ for some subset B of Y. Then

$f^{\leftarrow}(\chi_B) = \chi_A$, where $A = \{x \in X; f(x) \in B\}$,

the preimage of B. If φ is the characteristic function χ_A of an arbitrary subset A of X, then

$f_{\rightarrow}(\chi_A) = \chi_B$, where $B = \{f(x); x \in A\}$,

the direct image of A under f, sometimes written $f_{*}(A)$, and

$f^{\rightarrow}(\chi_A) = \chi_C$, where $C = Y \setminus \{f(x); x \notin A\}$, the complement of the direct image of the complement of A.

Because of this it is natural to define also a pullback $f^{\leftarrow} : \mathscr{P}(Y) \to \mathscr{P}(X)$ and pushforwards $f_{\to}, f^{\to} : \mathscr{P}(X) \to \mathscr{P}(Y)$, simply by identifying the power set $\mathscr{P}(X)$ with the set $\{0,1\}^X$ of characteristic functions.

There is in general no inclusion relation between the two sets B and C defined by $f_{\to}(\chi_A)$ and $f^{\to}(\chi_A)$, thus between $f_{\to}(A)$ and $f^{\to}(A) = \complement f_{\to}(\complement A)$. In fact, given any subdivision of Y into four disjoint subsets $Y_0 \cup Y_1 \cup Y_2 \cup Y_3$, we can easily find a mapping f and a subset A of X such that $f_{\to}(A) = Y_0 \cup Y_1$ and $f^{\to}(A) = Y_0 \cup Y_2$.

9. Structure of cleistomorphisms and anoiktomorphisms

The purpose of this section is to state and prove a structure theorem for anoiktomorphisms and cleistomorphisms in terms of quotients of mappings. The theorem has a certain formal resemblance to the result that any anoiktomorphism $\alpha : L \to L$ can be factorized as $\alpha = \delta \circ \varepsilon$ for a suitable erosion $\varepsilon : L \to M$ and a dilation $\delta : M \to L$ for some complete lattice M. One can take M here as the set of all α-open elements of L, a complete lattice, albeit in general not a sublattice of L (cf. Gierz et al. [14, p. 23, 3.13], [15, p. 29, O-3.13]).

However, we shall first present an easy result of the same kind in the special case of translation-invariant operators on the family of subsets of an abelian group.

Proposition 9.1. *Let S be a subset of an abelian group G. Then the dilation, erosion, cleistomorphism and anoiktomorphsim with structure element S can all be written in the form*

$$\varphi = (f/_{\star}g) \circ h,$$

where

$$f(B) = B \text{ or } B + S,\ g(B) = B + S,\ h(A) = A \text{ or } A + S, \quad A, B \in \mathscr{P}(G).$$

Proof. The dilation $\delta = \delta_S$, the erosion $\varepsilon = \delta_{[-1]}$, the cleistomorphism $\kappa = \varepsilon \circ \delta$, and the anoiktomorphism $\alpha = \delta \circ \varepsilon$ can be written

$$\delta(A) = \bigcup_B (B + S; B + S \subset A + S),$$
$$\varepsilon(A) = \bigcup_B (B; B + S \subset A),$$
$$\kappa(A) = \bigcup_B (B; B + S \subset A + S),$$
$$\alpha(A) = \bigcup_B (B + S; B + S \subset A).$$

We now let $f(B) = B + S, h(A) = A + S$ in the first case, $f(B) = B$, $h(A) = A$ in the second case, $f(B) = B, h(A) = A + S$ in the third case, and $f(B) = B + S, h(A) = A$ in the fourth case, while $g(B) = B + S$ in all four cases. $\square$

In the two last cases the result is valid generally:

Theorem 9.2. *Let $f : X \to M$ be any mapping from a set X into a complete lattice M. Then $\alpha = f/_{\star}f : M \to M$ is an anoiktomorphism. Conversely, any anoiktomorphism in M is of this form for some mapping $f : X \to M$ with $X = M$. By duality we get analogous statements for the upper quotient and cleistomorphisms.*

Proof. It is clear that $\alpha(y) = \bigvee(f(x); f(x) \leqslant y)$ defines a mapping $M \to M$ which is increasing and antiextensive. Idempotency remains to be proved. To do so we note that $f(x) \leqslant y$ if and only if $f(x) \leqslant \alpha(y)$. Therefore

$$\alpha(y) = \bigvee_x (f(x); f(x) \leqslant y) = \bigvee_x (f(x); f(x) \leqslant \alpha(y)) = \alpha(\alpha(y)).$$

Note that, by (7.5), $\alpha \circ f = f$, proving that the image of f is invariant under α; in other words, all elements in $\operatorname{im} f$ are α-open.

The converse follows from the formula $\alpha/_{\star}\alpha = \alpha$, which holds for any anoiktomorphism $\alpha : M \to M$. Indeed,

$$(\alpha/_{\star}\alpha)(y) = \bigvee_x (\alpha(x); \alpha(x) \leqslant y) = \bigvee_x (\alpha(x); \alpha(x) \leqslant \alpha(y)) = \alpha(y).$$

(Writing out the formula in full if $\alpha = f/_{\star}f$, we obtain $(f/_{\star}f)/_{\star}(f/_{\star}f) = f/_{\star}f = (f/_{\star}f) \circ (f/_{\star}f)$.) $\square$

Remark 9.3. We note that the choice of the mapping f in Proposition 9.1, taking $f(B) = B + S = \delta_S(B)$, is not available in the converse part of Theorem 9.2; instead, the mapping α itself serves as f.

10. Conclusion

We have introduced the notions of lower and upper inverses and lower and upper quotients for mappings between complete lattices—in fact, we have generalized the inverses to a more general setting. Their most basic properties have been investigated, in particular how the inverses can serve as left and right inverses to a given mapping. Important morphological operators can be systematically treated in the calculus created. In particular, anoiktomorphisms are always lower quotients of the form $f/_{\star}f$, and cleistomorphisms are always upper quotients of the form $f/^{\star}f$.

Acknowledgements

I am grateful to Gerald Banon for his careful comments on my paper (2007), which led to improvements in the presentation of the present paper. I also thank two anonymous referees for criticism enabling me to improve the paper.

References

[1] Gerald Jean Francis Banon, Junior Barrera, Decomposition of mappings between complete lattices by mathematical morphology, Signal Process. 30 (1993) 299–327.
[2] Garrett Birkhoff, Lattice Theory, American Mathematical Society, New York, NY, 1940. v + 155 pp.
[3] Garrett Birkhoff, Lattice Theory, vol. 25, third edition, eighth printing, American Mathematical Society Colloquium Publications, Providence, RI, 1995, xiii + 420 pp.
[4] Isabelle Bloch, Henri Maître, Fuzzy mathematical morphologies: a comparative study, Pattern Recognition 28 (9) (1995) 1341–1387.
[5] T.S. Blyth, Lattices and Ordered Algebraic Structures, Springer, 2005. X + 303 pp.
[6] T.S. Blyth, M.F. Janowitz, Residuation Theory, Pergamon Press, Oxford & al., 1972. ix + 379 pp.
[7] Jan Boman, Differentiability of a function and of its compositions with functions of one variable, Math. Scand. 20 (1967) 249–268.
[8] B.A. Davey, H.A. Priestely, Introduction to Lattices and Order, Cambridge University Press, Cambridge, 2003.
[9] T.-Q. Deng, Fuzzy logic and Mathematical Morphology, Report PNA-R0014, CWI, Amsterdam, 2000.
[10] T.-Q. Deng, H.J.A.M. Heijmans, Grey-scale morphology based on fuzzy logic, Report PNA-R0012, CWI, Amsterdam, 2000.
[11] P. Dubreil, M.L. Dubreil-Jacotin, Leçons d'algèbre moderne, second ed., Dunod, Paris, 1964.
[12] Werner Fenchel, On conjugate convex functions, Canad. J. Math. 1 (1949) 73–77.
[13] Bernhard Ganter, Rudolf Wille, Formal Concept Analysis: Mathematical Foundations, Springer-Verlag, Berlin & al., 1999. x + 284 pp.
[14] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer-Verlag, Berlin & al., 1980. XX + 371 pp.
[15] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, Continuous Lattices and Domains, Cambridge University Press, Cambridge, 2003. xxxvi + 591 pp.
[16] G. Grätzer, General Lattice Theory, Birkhäuser Verlag, Basel, 1998, xx + 663 pp (second printing 2003).

[17] H.J.A.M. Heijmans, Morphological Image Operators, Academic Press, Boston, 1994. xv + 509 pp.
[18] Henk J.A.M. Heijmans, Petros Maragos, Lattice calculus of the morphological slope transform, Signal Process. 59 (1997) 17–42.
[19] H.J.A.M. Heijmans, Ilya S. Molchanov, Morphology on convolution lattices with applications to the slope transform and random set theory, J. Math. Imaging Vision 8 (3) (1998) 199–214.
[20] H.J.A.M. Heijmans, C. Ronse, The algebraic basis of mathematical morphology I. Dilations and erosions, Comput. Vision Graphics Image Process. 50 (1990) 245–295.
[21] A. Kaufmann, M.M. Gupta, Fuzzy Mathematical Models in Engineering and Managements Science, North-Holland, Amsterdam, 1988.
[22] Christer O. Kiselman, Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants, Bull. Soc. Math. France 97 (1969) 329–356.
[23] Christer O. Kiselman, Digital Geometry and Mathematical Morphology. Lecture Notes, Uppsala University, 2002, 85 pp. Available at: <www.math.uu/~kiselman>.
[24] Christer O. Kiselman, A semigroup of operators in convexity theory, Trans. Amer. Math. Soc. 354 (5) (2002) 2035–2053.
[25] Christer O. Kiselman, Division of mappings between complete lattices, in: G.J.F. Banon, J. Barrera, U. de Mendonça Braga-Neto (Eds.), Mathematical Morphology and Its Applications to Signal and Image Processing. Proceedings of the Eighth International Symposium on Mathematical Morphology, October 10–13, Rio de Janeiro, RJ, Brasil, 2007, MCT/INPE, São José dos Campos, SP, Brasil, pp. 27–38.
[26] Andreas Kriegl, Peter W. Michor, The convenient setting of global analysis, Mathematical Surveys and Monographs, 53, American Mathematical Society, Providence, RI, 1997.
[27] A. Kriegl, L.D. Nel, Convenient vector spaces of smooth functions, Math. Nachr. 147 (1990) 39–45.
[28] Georges Matheron, Random Sets and Integral Geometry, Wiley & Sons, New York, NY & al., 1975. xxv + 261 pp.
[29] Peter Michor, A convenient setting for differential geometry and global analysis, Cahiers Topologique Géom. Différentielle 25 (1) (1984) 63–109. No. 2, 113–178.
[30] Oystein Ore, Galois connexions, Trans. Amer. Math. Soc. 55 (3) (1944) 493–513.
[31] Eike Petermann, On a method of constructing categories, J. Pure Appl. Algebra 15 (1979) 271–281.
[32] Christian Ronse, Why mathematical morphology needs complete lattices, Signal Process. 21 (1990) 129–154.
[33] C. Ronse, H.J.A.M. Heijmans, The algebraic basis of mathematical morphology. II. Openings and closings, Comput. Vision Graphics Image Process. Image Understanding 54 (1) (1991) 74–97.
[34] C. Ronse, H.J.A.M. Heijmans, A lattice-theoretical framework for annular filters in morphological image processing, Appl. Algebra Eng. Commun. Comput. 9 (1998) 45–89.
[35] Jean Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.
[36] Jean Serra, Éléments de théorie pour l'optique morphologique, Thèse de Doctorat d'État, Université Pierre et Marie Curie, 1986.
[37] Jean Serra, Thickenings, Thinnings, Technical Report 39. Fontainebleau, École Nationale Supérieure des Mines de Paris, Centre de Morphologie Mathématique, 1987.
[38] Jean Serra, Introduction to morphological filters, in: J. Serra (Ed.), Image Analysis and Mathematical Morphology, Theoretical Advances, vol. 2, Academic Press, London & al., 1988. xvi + 411 pp.
[39] Jean Serra, A lattice approach to image segmentation, J. Math. Imaging Vision 24 (1) (2006) 83–130.
[40] Ivan Singer, Abstract Convex Analysis, John Wiley & Sons, New York & al., 1997. xix + 491 pp.
[41] Nataša Sladoje, On Analysis of Discrete Spatial Fuzzy Sets in 2 and 3 Dimensions. Doctoral Thesis, Acta Universitatis Agriculturae Sueciae No. 2005:112. Uppsala: Faculty of Forest Science, Swedish University of Agricultural Sciences, 2005.
[42] Viggo Stoltenberg-Hansen, Ingrid Lindström, Edward R. Griffor, Mathematical Theory of Domains, Cambridge University Press, Cambridge, 1994. xii + 349 pp.
[43] Thomas Strömberg, The operation of infimal convolution, Dissertationes Math. (Rozprawy Mat.) 352 (1996) 58 pp.
[44] Morgan Ward, R.P. Dilworth, Residuated lattices, Trans. Amer. Math. Soc. 45 (3) (1939) 335–354.