Fuzzy Sets and Fuzzy Techniques

Lecture 9 – Fuzzy numbers and fuzzy arithmetics

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Fuzzy Sets and Fuzzy Techniques

Outline

- Interval numbers
- Interval arithmetics
- Fuzzy numbers
- Fuzzy arithmetics
- A lattice of fuzzy numbers, MIN and MAX operators
- Fuzzy equations
- An example of fuzzy linear programming

Interval numbers

An interval number, representing an uncertain real number

\[ A = [a_1, a_2] = \{ x \mid a_1 \leq x \leq a_2, x \in \mathbb{R} \} \]

A real number is a special case: point interval or singleton

\[ a = [a, a] \]

For the interval \( A \) we define

- **Width**
  \[ w(A) = a_2 - a_1 \]

- **Magnitude**
  \[ |A| = \max(|a_1|, |a_2|) \]
  (not to confuse with cardinality)

- **Image**
  \[ A^- = [-a_2, -a_1] \]

- **Inverse**
  \[ A^{-1} = \left[ \frac{1}{a_2}, \frac{1}{a_1} \right] \]
Interval numbers

Equality
\[ A = B \iff a_1 = b_1 \text{ and } a_2 = b_2 \]

Inclusion
\[ A \subseteq B \iff b_1 \leq a_1 \leq a_2 \leq b_2 \]

Distance
\[ d(A, B) = \max(|a_1 - b_1|, |a_2 - b_2|) \]

Arithmetic operations on intervals

For closed intervals \( A = [a_1, a_2] \) and \( B = [b_1, b_2] \), the four arithmetic operations are defined as follows (equivalent with definition on previous slide)

\[
A + B = [a_1 + b_1, a_2 + b_2] \\
A - B = A + B^- = [a_1 - b_2, a_2 - b_1] \\
A \cdot B = [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \\
\quad \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)]
\]

and, if \( 0 \notin [b_1, b_2] \)
\[
A/B = A \cdot B^{-1} = [a_1, a_2] \cdot \left[ \frac{1}{b_2}, \frac{1}{b_1} \right] = \left[ \min\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_2}{b_1}, \frac{a_1}{b_2}\right), \\
\quad \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_2}{b_1}, \frac{a_1}{b_2}\right) \right]
\]

Arithmetic operations on intervals

For intervals \( A \) and \( B \), and operator \( * \in \{+, -, \cdot, /\} \)
we define
\[ A * B = \{ a * b \mid a \in A, b \in B \} \]

Division, \( A/B \), is not defined when \( 0 \in B \).

The result of an arithmetic operation on closed intervals is again a closed interval.

Arithmetic operations on intervals

Let \( 0 = [0, 0] \) and \( 1 = [1, 1] \)

1. \( A + B = B + A \)
2. \( A \cdot B = B \cdot A \) (commutativity)
3. \( (A + B) + C = A + (B + C) \)
4. \( (A \cdot B) \cdot C = A \cdot (B \cdot C) \) (associativity)
5. \( A + 0 = A = A + 0 \)
6. \( A = 1 \cdot A = A \cdot 1 \) (identity)
7. \( A \cdot (B + C) \subseteq A \cdot B + A \cdot C \) (subdistributivity)
8. \( (A + C) \cdot B = A \cdot B + C \cdot B \) (distributivity)

Furthermore, if \( A = [a, a] \), then \( a \cdot (B + C) = a \cdot B + a \cdot C \)

9. \( 0 \in A - A \) and \( 1 \in A/A \)
10. If \( A \subseteq E \) and \( B \subseteq F \), then:
    \[ A + B \subseteq E + F \]
    \[ A - B \subseteq E - F \]
    \[ A \cdot B \subseteq E \cdot F \]
    \[ A/B \subseteq E/F \] (inclusion monotonicity)
Arithmetic operations on intervals

Note:

(i) $A - A = A + A^- = [- (a_2 - a_1), a_2 - a_1] \neq [0, 0]$

(ii) $A/A = A \cdot A^- \neq [1, 1]$

However (from prev. slide)

$0 \in A - A$ and $1 \in A/A$

Fuzzy numbers and fuzzy intervals

A fuzzy number is a fuzzy set on $\mathbb{R}$

$$A : \mathbb{R} \rightarrow [0, 1]$$

such that

(i) $A$ is normal ($\text{height}(A) = 1$)

(ii) $A$ is a closed interval for all $\alpha \in (0, 1]$

(iii) The support of $A$, $\text{Supp}(A) = 0^+A$, is bounded

Since all $\alpha$-cuts are closed intervals, every fuzzy number is a convex fuzzy set.

The converse is not necessarily true.

Multi-level interval numbers

Remark:

A path to fuzzy numbers (and then further on to fuzzy sets) taken by Bojadziev and Bojadziev is via two- and multi-level interval numbers, where different parts of the interval have different plausibility.

We, however, already know about fuzzy sets...
Theorem (4.1)
Let $A \in \mathcal{F}(\mathbb{R})$. Then, $A$ is a fuzzy number iff there exists a closed interval $[a, b] \neq \emptyset$ such that

$$A(x) = \begin{cases} 1 & \text{for } x \in [a, b] \\ l(x) & \text{for } x \in (\infty, a) \\ r(x) & \text{for } x \in (b, \infty) \end{cases}$$

where $l : (\infty, a) \to [0, 1]$ is monotonic non-decreasing, continuous from the right, and $l(x) = 0$ for $x < \omega_1$ and $r : (b, \infty) \to [0, 1]$ is monotonic non-increasing, continuous from the left, and $r(x) = 0$ for $x > \omega_2$.

An example of a function that maps to the set of fuzzy numbers.

Given a finite fuzzy set $A$, its fuzzy cardinality $|\tilde{A}|$ is a fuzzy number defined on $\mathbb{N}$ by

$$|\tilde{A}|(\alpha|A|) = \alpha$$

Linguistic variables

When fuzzy numbers are connected to linguistic concepts, such as very small, small, medium, and so on, and interpreted in a particular context, the resulting constructs are usually called linguistic variables.

A linguistic variable is fully characterized by a quintuple $(v, T, X, g, m)$, in which $v$ is the name of the variable, $T$ is the set of linguistic terms of $v$ that refer to the base variable whose values range over a universal set $X$, $g$ is a syntactic rule (a grammar) for generating linguistic terms, and $m$ is a semantic rule that assigns to each linguistic term $t \in T$ its meaning, $m(t)$, which is a fuzzy set on $X$ (i.e., $m : T \rightarrow \mathcal{F}(X)$).
Moving from interval numbers, we can define arithmetics on fuzzy numbers based on two principles:

1. **Cutworthiness** (thanks to inclusion monotonicity of intervals)
   \[ (A \ast B) = \alpha A \ast \alpha B \]
   in combination with
   \[ A \ast B = \bigcup_{\alpha \in [0,1]} \alpha (A \ast B) \]

2. **or the extension principle**
   \[ (A \ast B)(z) = \sup_{z = x \ast y} \min\{A(x), B(y)\} \]

**Theorem (4.2)**

Let \(* \in \{+, -, \cdot, /\}\), and let \(A, B\) denote continuous fuzzy numbers. Then, the fuzzy set \(A \ast B\) defined by the extension principle (prev. slide) is a continuous fuzzy number.

**Lemma**

\[ (A \ast B)(z) = \sup_{z = x \ast y} \min\{A(x), B(y)\} \Rightarrow (A \ast B) = \alpha A \ast \alpha B \]

So the two definitions are equivalent for continuous fuzzy numbers. (The proof is built on continuity.)

**MIN and MAX operators**

\[
\begin{align*}
\text{MIN}(A, B)(z) &= \sup_{z = \min(x, y)} \min\{A(x), B(y)\}, \\
\text{MAX}(A, B)(z) &= \sup_{z = \max(x, y)} \min\{A(x), B(y)\}
\end{align*}
\]

Again, for continuous fuzzy numbers, this is equivalent with a definition based on cutworthiness.

\[
\begin{align*}
\alpha (\text{MIN}(A, B)) &= \text{MIN}(\alpha A, \alpha B), \\
\alpha (\text{MAX}(A, B)) &= \text{MAX}(\alpha A, \alpha B), \quad \forall \alpha \in (0,1].
\end{align*}
\]

Where, for intervals \([a_1, a_2], [b_1, b_2]\)

\[
\begin{align*}
\text{MIN}([a_1, a_2], [b_1, b_2]) &= \min\{a_1, b_1\}, \min\{a_2, b_2\}, \\
\text{MAX}([a_1, a_2], [b_1, b_2]) &= \max\{a_1, b_1\}, \max\{a_2, b_2\}.
\end{align*}
\]
MIN and MAX operators

Theorem (4.3)

- \( \text{MIN}(A, B) = \text{MIN}(B, A) \),
  \( \text{MAX}(A, B) = \text{MAX}(B, A) \) (commutativity)
- \( \text{MIN}[\text{MIN}(A, B), C] = \text{MIN}[A, \text{MIN}(B, C)] \),
  \( \text{MAX}[\text{MAX}(A, B), C] = \text{MAX}[A, \text{MAX}(B, C)] \) (associativity)
- \( \text{MIN}(A, A) = A \),
  \( \text{MAX}(A, A) = A \) (idempotence)
- \( \text{MIN}[\text{MIN}(A, B), \text{MAX}(A, B)] = A \),
  \( \text{MAX}[\text{MAX}(A, B), \text{MIN}(A, B)] = A \) (absorption)
- \( \text{MIN}[\text{MIN}(A, B), \text{MAX}(B, C)] = \text{MAX}[\text{MIN}(A, B), \text{MIN}(A, C)] \),
  \( \text{MAX}[\text{MAX}(A, B), \text{MIN}(A, B)] = \text{MIN}[\text{MAX}(A, B), \text{MAX}(A, C)] \) (distributivity)

Lattice of fuzzy numbers

Denoting the set of fuzzy numbers with \( \mathcal{R} \),
\( (\mathcal{R}, \text{MIN}, \text{MAX}) \) is a **distributive lattice**, in which MIN and MAX represent the meet and join, respectively.

Recall: A lattice is a **partially ordered set** (or poset) whose nonempty finite subsets all have a unique supremum (called join) and an infimum (called meet).

The lattice can also be expressed as \( (\mathcal{R}, \preceq) \), where \( \preceq \) is a **partial ordering** on \( \mathcal{R} \) given by

\[
A \preceq B \iff \text{MIN}(A, B) = A \text{ or, alternatively } A \preceq B \iff \text{MAX}(A, B) = B
\]

The partial order expressed in terms of \( \alpha \)-cuts

\[
A \preceq B \iff \text{MIN}(\alpha A, \alpha B) = \alpha A,
\]

\[
A \preceq B \iff \text{MAX}(\alpha A, \alpha B) = \alpha B,
\]

for all \( \alpha \in (0, 1] \).

Where, for intervals \([a_1, a_2], [b_1, b_2] \]

\[
[a_1, a_2] \preceq [b_1, b_2] \iff a_1 \leq b_1 \text{ and } a_2 \leq b_2
\]

Thus, for any \( A, B \in \mathcal{R} \), we have

\[
A \preceq B \text{ iff } \alpha A \preceq \alpha B
\]

for all \( \alpha \in (0, 1] \).
Lattice of fuzzy numbers

Not all fuzzy numbers are comparable (only partial order). However, values of linguistic variables are often defined by fuzzy numbers that are comparable.

For example:

very small \leq small \leq medium \leq large \leq very large

Interval equations

Equation $A + X = B$

Let $X = [x_1, x_2]$.

Then $[a_1 + x_1, a_2 + x_2] = [b_1, b_2]$ follows immediately.

Clearly: $x_1 = b_1 - a_1$ and $x_2 = b_2 - a_2$.

Since $X$ must be an interval, it is required that $x_1 \leq x_2$.

That is, the equation has a solution if $b_1 - a_1 \leq b_2 - a_2$.

Then $X = [b_1 - a_1, b_2 - a_2]$ is the solution.

Fuzzy equations

The solution to a fuzzy equation can be obtained by solving a set of interval equations, one for each nonzero $\alpha$ in the level set $\Lambda(A) \cup \Lambda(B)$.

The equation $A + X = B$ has a solution iff

(i) $^{\alpha}b_1 - ^{\alpha}a_1 \leq ^{\alpha}b_2 - ^{\alpha}a_2$ for every $\alpha \in (0, 1]$, and

(ii) $\alpha \leq \beta$ implies $^{\alpha}b_1 - ^{\alpha}a_1 \leq ^{\beta}b_1 - ^{\beta}a_1 \leq ^{\beta}b_2 - ^{\beta}a_2 \leq ^{\alpha}b_2 - ^{\alpha}a_2$.

If a solution $^{\alpha}X$ exists for every $\alpha \in (0, 1]$ (property (i)), and property (ii) is satisfied, then the solution $X$ is given by

$X = \bigcup_{\alpha \in (0, 1]} ^{\alpha}X$
Similarly as $A + X = B$

The equation $A \cdot X = B$ has a solution iff

(i) $a_{ij}/a_i \leq c_{ij}/c_i$ for every $a \in [0, 1]$, and

(ii) $\alpha \leq \beta$ implies $a_{ij}/a_i \leq \beta_{ij}/\beta_i \leq \beta_{ij}/\beta_i \leq \alpha_{ij}/\alpha_i$.

If the solution exists, it has the form

$$X = \bigcup_{\alpha \in [0, 1]} \alpha X$$

where $\alpha X = [a_{ij}/a_i, \alpha b_{i}/\alpha a_i]$. Again, $X = B/A$ is not a solution of the equation.

### Linear programming

The classical linear programming problem is to find the minimum or maximum values of a linear function under constraints represented by linear inequalities or equations.

The most typical linear programming problem is:

Minimize (or maximize) $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2$$
$$\ldots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m$$

$x_1, x_2, \ldots, x_n \geq 0$.

### An example

Minimize $z = x_1 - 2x_2$

Subject to

$$3x_1 - x_2 \geq 1$$
$$2x_1 + x_2 \leq 6$$
$$0 \leq x_2 \leq 2$$
$$0 \leq x_1$$

The feasible set, i.e., the set of vectors $x$ that satisfy all constraints, is always a convex polygon (if bounded).

Figure: An example of a classical linear programming problem.
In many practical situations, it is not reasonable to require that the constraints or the objective function are specified in crisp precise terms. The most general case of fuzzy linear programming grows rather complex, and is not discussed in the book. A realistic special case to provide the feeling: The situation where the right-hand-side vector and the constraint matrix are expressed by fuzzy triangular numbers.

Any triangular fuzzy number can be represented by three real numbers, $s$, $l$, $r$.

\[
\text{Minimize } z = \sum_{j=1}^{n} c_j x_j \\
\text{Subject to } \sum_{j=1}^{n} \langle s_{ij}, l_{ij}, r_{ij} \rangle x_j \leq \langle t_i, u_i, v_i \rangle, \ i = 1 \ldots m \\
x_j \geq 0, \ j = 1 \ldots n
\]

Where $A_{ij} = \langle s_{ij}, l_{ij}, r_{ij} \rangle$, $B_i = \langle t_i, u_i, v_i \rangle$ and $\leq$ is defined by the partial order $\preceq$, i.e., $A \preceq B$ iff $\text{MAX}(A, B) = B$. 

**Figure:** Triangular fuzzy number, represented by the triple $(s, l, r)$.
Fuzzy linear programming

For any two triangular numbers, \( A = (s_1, l_1, r_1) \) and \( B = (s_2, l_2, r_2) \),

\[
A \leq B \text{ iff } \begin{cases} 
  s_1 \leq s_2, \\
  s_1 - l_1 \leq s_2 - l_2 \\
  s_1 + r_1 \leq s_2 + r_2
\end{cases}
\]

Moreover \( (s_1, l_1, r_1) + (s_2, l_2, r_2) = (s_1 + s_2, l_1 + l_2, r_1 + r_2) \) and \( (s_1, l_1, r_1)x = (s_1x, l_1x, r_1x) \) for any non-negative real number \( x \).

This enables us to rewrite the problem as follows:

Minimize \( z = \sum_{j=1}^{n} c_j x_j \)

Subject to \( \sum_{j=1}^{n} s_{ij} x_j \leq t_i \), \( i = 1 \ldots m \)

\( \sum_{j=1}^{n} (s_{ij} - l_{ij}) x_j \leq t_i - u_i \), \( i = 1 \ldots m \)

\( \sum_{j=1}^{n} (s_{ij} + r_{ij}) x_j \leq t_i + v_i \), \( i = 1 \ldots m \)

\( x_j \geq 0 \), \( j = 1 \ldots n \)

which is a classic (crisp) linear programming problem that we know how to solve :-)

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