What is a fuzzy set?

A set is a collection of its members.

The notion of fuzzy sets is an extension of the most fundamental property of sets. Fuzzy sets allows a grading of to what extent an element of a set belongs to that specific set.
Why Fuzzy?

Precision is not truth.
- Henri Matisse

So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.
- Albert Einstein

As complexity rises, precise statements lose meaning and meaningful statements lose precision.
- Lotfi Zadeh

What is a fuzzy set?

Randomness vs. Fuzziness

Randomness refers to an event that may or may not occur. **Randomness:** frequency of car accidents.

Fuzziness refers to the boundary of a set that is not precise. **Fuzziness:** seriousness of a car accident.

Prof. George J. Kli

Fuzzy is not just another name for probability.

The number 10 is **not** probably big!
...and number 2 is **not** probably not big.

Uncertainty is a consequence of non-sharp boundaries between the notions/objects, and not caused by lack of information.

Statistical models deal with random events and outcomes; fuzzy models attempt to capture and quantify nonrandom imprecision.

An example

A fuzzy set of tall men

<table>
<thead>
<tr>
<th>Name</th>
<th>Height, cm</th>
<th>Degree of Membership</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chris</td>
<td>208</td>
<td>1.00</td>
</tr>
<tr>
<td>Mark</td>
<td>205</td>
<td>1.00</td>
</tr>
<tr>
<td>John</td>
<td>198</td>
<td>0.98</td>
</tr>
<tr>
<td>Tom</td>
<td>181</td>
<td>0.82</td>
</tr>
<tr>
<td>David</td>
<td>179</td>
<td>0.78</td>
</tr>
<tr>
<td>Mike</td>
<td>172</td>
<td>0.24</td>
</tr>
<tr>
<td>Bob</td>
<td>167</td>
<td>0.15</td>
</tr>
<tr>
<td>Steven</td>
<td>158</td>
<td>0.06</td>
</tr>
<tr>
<td>Bill</td>
<td>155</td>
<td>0.01</td>
</tr>
<tr>
<td>Peter</td>
<td>132</td>
<td>0.00</td>
</tr>
</tbody>
</table>

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Another example

Age groups

Computing with Words

- Lexical Imprecision
- Linguistic Variable

Yet another example

Temperature

Medical Diagnosis

Medicine is one field in which the applicability of fuzzy set theory (FST) was recognized quite early (mid-1970s). Diagnosis of disease has frequently been the focus of application of FST.

The process of classifying different sets of symptoms under a single name and determining appropriate therapeutic actions becomes increasingly difficult.
The best and most useful descriptions of diseases entities often use linguistic terms that are irreducibly vague.

**Example:** Hepatitis

"Total proteins are usually normal, albumin is decreased, alpha-globulin are slightly decreased, beta-globulins are slightly decreased, and gamma-globulins are increased."

The linguistic terms printed in blue color are inherently vague.

Patients suffering from hepatitis show in 60% of all cases high fever, in 45% of all cases a yellowish colored skin, and in 30% of all cases nausea.
Aristotle introduced the laws of thought which consisted of three fundamental laws:

- principle of identity
- law of the excluded middle
- law of contradiction

The law of the excluded middle states that for all propositions \( p \), either \( p \) or \( \neg p \) must be true, there being no middle true proposition between them. In other words, \( p \) cannot be both \( p \) and not \( p \).

Plato laid the foundation of what is now known as fuzzy logic indicating that there was a third region beyond true and false. It was Jan Łukasiewicz (in 1910) who first proposed a systematic alternative to the bi-valued logic of Aristotle and described the 3-valued logic, with the third value being Possible.

Lotfi Zadeh, in his theory of fuzzy logic, proposed the making of the membership function operate over the range of real numbers \([0,1]\). He proposed new operations for the calculus of logic and showed that fuzzy logic was a generalization of classical logic.

Who knows? Zadeh is too busy pushing forward to keep up with how far the field has expanded. His office in the newly constructed Computer Science Building at Berkeley is stacked floor to ceiling with reprints of articles related to Fuzzy. He believes that people are studying this field in every country which offers advanced education. Twelve journals are now published which include the word “Fuzzy” in their title. An estimated 15,000 articles have been published, although it’s hard to be exact as some appear in obscure journals in remote parts of the world. An estimated 3,000 patents have been applied for and 1,000 granted. The Japanese, with 2,000 scientists involved in Fuzzy Logic, have been very quick to incorporate Fuzzy Logic in the design of consumer products, such as household appliances and electronic equipment and one company, Mitsushita (which sells under the name of Panasonic and Quasar) acknowledged that in 1991-1992 alone, they had sold more than 1 billion dollars worth of equipment that used Fuzzy Logic. The concept is so popular there that the English word has entered the Japanese language, though the Japanese pronounce it more like “fudgy” than “fuzzy”.

Short Biographical Sketch, Azerbaijan International 1994, by Betty Blair

Google Scholar (2010-02-14): “Fuzzy sets, 1965” - Cited by 22411

L. Zadeh Fuzzy Logic 40 Years Later


http://www.cb.uu.se/~joakim/course/fuzzy/vt10/

- 13 lectures + repetition
- 2 computer exercises
- 1 small project work + presentation
- Written exam

What will we learn in this course?

- The basics of fuzzy sets
  - How to define fuzzy sets
  - How to perform operations on fuzzy sets
  - How to extend crisp concepts to fuzzy ones
  - How to extract information from fuzzy sets
- The very basics of fuzzy logic and fuzzy reasoning
- We will look at some applications of fuzzy in
  - Image processing
  - Control systems
  - Machine intelligence / expert systems

Teachers

- Joakim Lindblad
- Nataša Sladoje
- Milan Gavrilovic (labs)

Computer exercises

- Something simple just to make sure that you are following
  - Fuzzy thresholding and clustering
  - Fuzzy morphology
Project work
Apply fuzzy in your own work

- Groups of two
- Compare with traditional (crisp)
- 15 min. presentation (16th of March)

Exam
CBA 19th of March

- Will not be too hairy... but not all to easy either.

Course Literature
The book

http://www.cb.uu.se/~joakim/course/fuzzy/vt10/literature.html

Fuzzy Sets and Fuzzy Logic: Theory and Applications
- Covers more than the course
- Emphasis on theory
  + Good and reliable
  - Hard to find
  - Expensive

Fuzzy Algorithms: With Applications to Image Processing and Pattern Recognition
- Will be used toward the end of the course (applications)
Chapter one...

From Ordinary (Crisp) Sets to Fuzzy Sets
A Grand Paradigm Shift

At the beginning of his work Beiträge zur Begründung der transfiniten Mengenlehre, Georg Cantor, the principal creator of set theory, made the following definition of a set:

“By a set we understand any collection $M$ of definite, distinct objects $m$ of our perception or of our thought (which will be called the elements of $M$) into a whole.”

The objects of a set are also called its members. The elements of a set can be anything: numbers, people, letters of the alphabet, other sets, and so on.

1.2 Crisp sets: An overview

- Sets: $\mathbb{Z}$, $\mathbb{R}$, intervals, ordered pairs
- Notation: member, $\in$, $\notin$
- List, rule, characteristic fun.
- Set of sets = family of sets
- Subset ($\subseteq$), equality ($=$), inequality ($\neq$), proper subset ($\subset$)
- Power set $\mathcal{P}$ (higher orders)
- Cardinality ($|\cdot|$)
- Relative complement ($B \setminus A$) (a.k.a. set difference)
- Universal set
- (Absolute) complement (is involutive, i.e. bijective and symmetric)
- Union ($\cup$), intersection ($\cap$)
Fuzzy sets

A fuzzy set of a reference set is a set of ordered pairs

\[ F = \{ (x, \mu_F(x)) \mid x \in X \}, \]

where \( \mu_F : X \rightarrow [0, 1] \).

Where there is no risk for confusion, we use the same symbol for the fuzzy set, as for its membership function. Thus

\[ F = \{ (x, F(x)) \mid x \in X \}, \]

where \( F : X \rightarrow [0, 1] \).

To define a fuzzy set \( \iff \) To define a membership function

**Continuous (analog) fuzzy sets**

\[ A : X \rightarrow [0, 1], \ X \text{ is dense} \]

**Discrete fuzzy sets**

\[ A : \{ x_1, x_2, x_3, \ldots, x_s \} \rightarrow [0, 1] \]

**Digital fuzzy sets**

If a discrete-universal membership function can take only a finite number \( n \geq 2 \) of distinct values, then we call this fuzzy set a digital fuzzy set.

\[ A : \{ x_1, x_2, x_3, \ldots, x_s \} \rightarrow \{ 0, \frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, \ldots, \frac{n-2}{n-1}, 1 \} \]

**List form**

\[ M = \{ (1, 1), (2, 1), (3, 0.9), (4, 0.7), (5, 0.3), \ldots \} \]

Note: The list form can be used only for finite sets.

**Rule form**

\[ M = \{ x \in X \mid x \text{ meets some conditions} \}, \]

where the symbol \( \mid \) denotes the phrase "such that".

**Membership form**

Let \( M \) be a fuzzy set named "numbers closed to zero"

\[ M(x) = e^{-x^2} \text{ for } x \in [-3, 3] \]

\[ M(0) = 1, M(2) = e^{-4}, M(-2) = e^{-4} \]
Fuzzy Sets
and Fuzzy
Techniques
Joakim
Lindblad

Outline
Introduction
About the course
Chapter one (Crisp) Set Theory
1.2 Crisp sets: An overview
Fuzzy sets
Fuzzy sets of different types
Fuzzy sets of different levels
Basic concepts and terminology
1.3 Fuzzy sets: Basic types
1.4 Fuzzy sets: Basic concepts

Fuzzy Sets
and Fuzzy
Techniques
Joakim
Lindblad

Fuzzy sets

Also the domain of the membership function may be fuzzy. Fuzzy sets defined so that the elements of the universal set are themselves fuzzy sets are called level 2 fuzzy sets.

The membership function may be vague in itself.

**Fuzzy sets of different types**

The membership function may be vague in itself.

**Interval-valued fuzzy sets**

\[ A : X \rightarrow \mathcal{E}([0, 1]) \]

**Fuzzy sets of type 2**

\[ A : X \rightarrow \mathcal{F}([0, 1]) \]

**L-fuzzy sets** – \( L \) is any partially ordered set

\[ A : X \rightarrow L \]

**Basic concepts and terminology**

The support of a fuzzy set \( A \) in the universal set \( X \) is a crisp set that contains all the elements of \( X \) that have nonzero membership values in \( A \), that is,

\[ \text{supp}(A) = \{ x \in X \mid A(x) > 0 \} \]

A fuzzy singleton is a fuzzy set whose support is a single point in \( X \).
Basic concepts and terminology

A **crossover point** of a fuzzy set is a point in X whose membership value to A is equal to 0.5.

The **height**, \( h(A) \) of a fuzzy set \( A \) is the largest membership value attained by any point. If the height of a fuzzy set is equal to one, it is called a **normal** fuzzy set, otherwise it is subnormal.

We observe that the strong \( \alpha \)-cut \( 0^+A \) is equivalent to the support \( \text{supp}(A) \).

The 1-cut \( 1A \) is often called the **core** of \( A \).

**Note!** Sometimes the highest non-empty \( \alpha \)-cut \( h(A)A \) is called the core of \( A \). (in the case of subnormal fuzzy sets, this is different).

The word **kernel** is also used for both of the above definitions. (Total confusion!)

An **\( \alpha \)-cut** of a fuzzy set \( A \) is a **crisp set** \( \alpha A \) that contains all the elements in X that have membership value in \( A \) greater than or equal to \( \alpha \).

\[
\alpha A = \{ x | A(x) \geq \alpha \}
\]

A **strong **\( \alpha \)-cut of a fuzzy set \( A \) is a crisp set \( \alpha^+A \) that contains all the elements in \( X \) that have membership value in \( A \) strictly greater than \( \alpha \).

\[
\alpha^+A = \{ x | A(x) > \alpha \}
\]

The ordering of the values of \( \alpha \) in \([0,1]\) is **inversely** preserved by set inclusion of the corresponding \( \alpha \)-cuts as well as strong \( \alpha \)-cuts. That is, for any fuzzy set \( A \) and \( \alpha_1 < \alpha_2 \) it holds that

\[
\alpha_2 A \subseteq \alpha_1 A.
\]

All \( \alpha \)-cuts and all strong \( \alpha \)-cuts form two distinct families of **nested** crisp sets.

The set of all levels \( \alpha \in [0,1] \) that represent distinct \( \alpha \)-cuts of a given fuzzy set \( A \) is called the **level set** of \( A \).

\[
\Lambda(A) = \{ \alpha | A(x) = \alpha \text{ for some } x \in X \}.
\]
Basic concepts and terminology

A fuzzy set \( A \) defined on \( \mathbb{R}^n \) is **convex** iff

\[
A(\lambda x_1 + (1 - \lambda)x_2) \geq \min (A(x_1), A(x_2)),
\]

for all \( \lambda \in [0,1] \), \( x_1, x_2 \in \mathbb{R}^n \) and all \( \alpha \in [0,1] \).

Or, equivalently, \( A \) is **convex** if and only if all its \( \alpha \)-cuts \( ^\alpha A \), for any \( \alpha \) in the interval \( \alpha \in (0,1] \), are convex sets.

Any property that is generalized from classical set theory into the domain of fuzzy set theory by requiring that it holds in all \( \alpha \)-cuts in the classical sense is called a **cutworthy** property.

1.4 Fuzzy sets: Basic concepts

- **alpha-cut and strong alpha-cut**
- The level set \( \Lambda \) (subset range \([0,1]\))
- Subset ordering of alpha cuts (inverse to alpha) \( \rightarrow \) nested crisp sets
- New way to define FS, set of alpha cuts
- Support, core, height, normal & subnormal
- Convexity (on \( \mathbb{R}^n \)) (not convex function!) [proof]
- Cutworthy and strong cutworthy properties (holds for all alpha cuts)