Fuzzy Sets and Fuzzy Techniques
Lecture 13 – Defuzzification

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Literature


Definition

Defuzzification is a process that maps a fuzzy set to a crisp set.

Defuzzification has attracted far less attention than other processes involved in fuzzy systems and technologies.

- Defuzzification is sometimes not seen as a part of the core of a fuzzy system; fuzzy system “ends” where uncertainty and imprecision end.
- Defuzzification is “just the last step”; it is an interface with crisp models of the world.
- Defuzzification is a model of synthesis, therefore it is completely opposite to the main concept of fuzzy set theory.
Approaches

- Defuzzification to a point.
- Defuzzification to a set.
- Generating a good representative of a fuzzy set.
- Recovering a crisp original set.

Criteria

**Defuzzification to a point**

- Arbitrary universes
  - Core selection criterion
  - Scale invariance (ordinal, interval, ratio, relative, absolute)
- Ordered universes
  - Monotony
  - \( t \)-conorm criterion
- The set of real numbers
  - \( x \)-Translation
  - \( x \)-Scaling
  - Continuity

**Defuzzification to a set**

- Defuzzification should contain the kernel of a fuzzy set.
- Defuzzification should be consistent with affine transformations.
- Defuzzification should preserve a natural order in the set of fuzzy sets.
  
  If \( F \leq G \), then \( D(F) \subseteq D(G) \), for fuzzy sets \( F, G \) and their averagings \( D(F) \) and \( D(G) \).

**A list of methods**

- Maxima methods and derivatives
  Selection of an element from the core of a fuzzy set as defuzzification value. Main advantage is simplicity.
- Distribution methods and derivatives
  Conversion of the membership function into a probability distribution, and computation of the expected value. Main advantage is continuity property.
- Area methods
  The defuzzification value divides the area under the membership function in two (more or less) equal parts.
Defuzzification to a point
A list of methods – Maxima methods

- Random choice of maxima

\[ \text{Prob}(D(A) = x_0) = \begin{cases} \frac{1}{|\text{core}(A)|} & x_0 \in \text{core}(A), \\ 0 & \text{otherwise} \end{cases} \]

- First of maxima, Last of maxima, Middle of maxima

\[
\begin{align*}
FOM(A) &= \min \text{core}(A) \\
LOM(A) &= \max \text{core}(A)
\end{align*}
\]

For \( \text{MOM}(A) \) it holds that

\[
|\text{core}(A)_{< \text{MOM}(A)}| = |\text{core}(A)_{> \text{MOM}(A)}| \quad \text{if} \quad |\text{core}(A)| \quad \text{is odd}
\]

\[
|\text{core}(A)_{< \text{MOM}(A)}| = |\text{core}(A)_{> \text{MOM}(A)}| \pm 1 \quad \text{if} \quad |\text{core}(A)| \quad \text{is even.}
\]

Defuzzification to a point
A list of methods – Distribution methods (1)

- Centre of gravity (Set of real numbers)

\[
\text{COG}(A) = \frac{\sum_{x_{\text{max}}} x \cdot A(x)}{\sum_{x_{\text{max}}} A(x)}
\]

- Mean of maxima (Set of real numbers)

\[
\text{MeOM}(A) = \frac{\sum_{x \in \text{core}(A)} x}{|\text{core}(A)|}.
\]

- Basic defuzzification distribution

\[
\text{BADD}(A, \gamma) = \frac{\sum_{x_{\text{max}}} x \cdot A(x)^{\gamma}}{\sum_{x_{\text{max}}} A(x)^{\gamma}}
\]

where \( \gamma \in [0, \infty) \) reflects the confidence in the system.

Defuzzification to a point
A list of methods – Distribution methods (2)

- Generalized level set defuzzification

\[
\text{GLSD}(A, \gamma) = \frac{\sum_{i=1}^{N} c_i m_i \gamma^i}{\sum_{i=1}^{N} c_i \gamma^i}
\]

where \( \gamma \in (0, \infty) \) is the confidence to the system, \( N \) is the number of \( \alpha \)-cuts, \( c_i = |A_{\alpha_i}| \), and \( m_i \) is the average value of the \( i \)-th \( \alpha \)-cut.

- Indexed centre of gravity

\[
\text{ICOG}(A) = \frac{\sum_{x \in A_{\alpha}} x \cdot A(x)}{\sum_{x \in A_{\alpha}} A(x)}
\]

where \( \alpha \) is a selected threshold below which all membership values are set to zero.

- Fuzzy mean (combines aggregation and defuzzification)

\[
\text{FM}(A) = \frac{\sum_{i=1}^{N} \alpha_i |A_{\alpha_i}|}{\sum_{i=1}^{N} \alpha_i}
\]

where \( N_{\alpha} \) is the number of fuzzy output sets, \( \alpha_i \) is the degree of consistency obtained by inference rules, and \( \alpha_i \) is some numerical value associated with the output fuzzy sets \( A_i \) (often \( \alpha_i = \text{MOM}(A_i) \)).
Defuzzification to a point
Criteria fulfilment

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Defuzzification to a set
Averaging procedures

- \( \alpha \)-cuts chosen at various levels \( \alpha \).
- Average \( \alpha \)-cuts based on an integration of set-valued function, called Kudo-Aumann integration.

Example:
The separating power of a fuzzy set \( F \) is a set \( Z \subset \text{supp}(F) \), with a compact support and non-zero Lebesgue measure \( (\mu(\text{supp}(Z)) \neq 0) \), such that the function

\[
\Theta_F(Z) = \frac{\int_Z F(v) dv}{\mu(Z)} - \frac{\int_Z F(v) dv}{\mu(Z)}
\]

attains its supremum for it. \( \bar{Z} \) denotes the complement of \( Z \) with respect to the support of \( F \).

\( SP(F) \) is an averaging procedure.
Defuzzification to a set
Averaging procedures

Possibly better choice is shown below: $M(F) = \frac{1}{2}(A_1 + A_\frac{1}{2})$.

Defuzzification to a set
Average $\alpha$-cuts

Let a fuzzy set $A$ be given by a membership function $\mu : R \rightarrow [0, 1]$.
- Sets $F(w)$ are $\alpha$-cuts, $A_\alpha$, of the fuzzy set $A$, for $\alpha \in [0, 1]$;
- Selectors are $\varphi(\alpha) = \inf A_\alpha$ and $\phi(\alpha) = \sup A_\alpha$.

Then, the average $\alpha$-cut of $A$ is

$$A_\mu = \left[\int_{[0,1]} \inf A_\alpha \, d\alpha, \int_{[0,1]} \sup A_\alpha \, d\alpha\right].$$

Defuzzification to a set
Average $\alpha$-cuts

For a set valued function $F : \Omega \rightarrow \mathcal{P}(R^n)$, Kudo-Aumann integral is defined as

$$\int_{\Omega} F \, dm = \left\{ \int_{\Omega} f \, dm \mid f \in S(F) \right\}$$

where

$$S(F) = \{ f \mid f(w) \in F(w) \ a.e., f \text{ is integrable} \}$$

is the space of integrable selectors of $F$.

Note that the integral of a set-valued function is a set.

Defuzzification to a set
Average $\alpha$-cuts

Often, $A_\mu = A_{0.5}$.

Not always.

Example:

$$\mu(x) = \begin{cases} 4(x - x^2), & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

In this case, $A_\mu = [\frac{1}{4}, \frac{1}{2}]$.

$A_\mu$ is not always an $\alpha$-cut of $A$, for any $\alpha$.

Example:

$$\mu(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 4(x - x^2), & x \in (\frac{1}{2}, 1] \\ 0, & \text{otherwise} \end{cases}$$

In this case, $A_\mu = [\frac{1}{4}, \frac{1}{2}]$. Since $\mu(\frac{1}{2}) \neq \mu(\frac{5}{8})$, $A_\mu \neq A_\alpha$ for every $\alpha \in [0, 1]$.
Defuzzification respecting the fuzzification

What is the idea?

Assume:
- universal set being the set of real numbers;
- all parameters of the fuzzification are known;
- a value \( y \in R \) is fuzzified and a vector \( v \), characterizing \( y \) by fuzzy membership values corresponding to each of the linguistic terms used, is known.

Define defuzzification so that \( y \) is recovered from \( v \).

Defuzzification respecting the fuzzification

How does it look in reality?

Constraints:
- All the parameters of the fuzzification are known.
- The value \( v \) is such that \( F(y_0) = v \) for some \( y_0 \).

In reality, given \( v \) is rarely such, i.e., there is no \( y_0 \) fulfilling \( F(y_0) = v \).

Instead, the fuzzy representation of a “candidate” for \( y_0 \) is generated, using \( v \) and fuzzy sets representing linguistic terms (i.e., using knowledge about fuzzification).

To obtain a crisp solution \( y_0 \), defuzzification (to a point) of its fuzzy representation is performed.

Example:

Let 4 linguistic terms be represented by fuzzy sets \( f_i \) as shown:

\[ \nu = (0; 0.2; 0.8; 0) \]

is the vector of fuzzy membership values corresponding to \( y \).

Partial solutions of the problem are the sets \( Y_i \), containing elements \( y_0 \) such that \( f_i(y_0) = \nu_i \), for each \( i = 1, 2, 3, 4 \).

In this case, \( Y_1 = [1, 3] \), \( Y_2 = \{0.2, 1.8\} \), \( Y_3 = \{1.8, 2.2\} \), \( Y_4 = [0, 2] \).

The solution is defined as the intersection of all partial solutions:

\[ Y_f = \bigcap_{i=1}^4 Y_i \]

and in this case \( Y_f = 1.8 \) is obtained.

A bit more general approach

In case when for a given \( v \)

\[ v \neq F(y), \quad \text{for all } y, \]

a “candidate” for \( y \) is found by an optimization (minimization) of the function:

\[ J = \frac{1}{2} \sum_{i=1}^n (\nu_i - f_i(y))^2. \]
Assume:

- Functions $f_i$ (used for fuzzification) are triangular. The core of each $f_i$ is a singleton, denoted by $C_i$.
- For a given $v = (v_1, v_2, \ldots, v_n)$, and known $(C_1, C_2, \ldots, C_n)$, we look for $y_0$ such that $y_0$ is equal to $C_i$ with a weight of $v_i$.

In other words, $y_0$ minimizes

$$J = \sum_{i=1}^{n} v_i \cdot D(y_0, C_i) = \sum_{i=1}^{n} v_i (y_0 - C_i)^2.$$

Constraints on fuzzification:

- For all $y$, there are at most two $f_i$ (linguistic terms) such that $f_i(y) > 0$.
- The kernel $C_i$ of each $f_i$ is a singleton.
- $f_i$ are monotonically increasing for $x < C_i$ and monotonically decreasing for $x > C_i$.
- The sum of membership values of each element $y$ to all the fuzzy sets $f_i$ is equal to 1.
- All $f_i$ have the same shape (are defined by the same monotonically increasing function $f$, s.t. $f(0) = 0$ and $f(1) = 1$).

Analytical solution leads to defuzzification by the **height method**, which computes the barycentre of membership function kernels, weighted by the corresponding membership degrees:

$$y_0 = \frac{\sum_{i=1}^{n} v_i C_i}{\sum_{i=1}^{n} v_i}.$$

Recall that the fuzzification assumed is based on triangular membership functions.

The aim is to find $y_0$ such that

$$J = \sum_{i=1}^{n} H(v_i) \cdot D(y_0, C_i) = \sum_{i=1}^{n} H(v_i)(y_0 - C_i)^2,$$

where the function $H : [0, 1] \rightarrow [0, 1]$ appropriately adjusts the weights $v_i$ in the case when non-linear fuzzification is applied, and is, under the assumptions above, shown to be equal to $f^{-1}$.

The defuzzification method that is in this way developed is

$$y = \frac{\sum_{i=1}^{n} f^{-1}(v_i) C_i}{\sum_{i=1}^{n} f^{-1}(v_i)}.$$
Defuzzification by feature distance minimization

Literature: