Fuzzy Sets and Fuzzy Techniques
Lecture 8 – Operations on Fuzzy Sets

Nataša Sladoje

Centre for Image Analysis
Uppsala University

February 15, 2007

Standard fuzzy operations

Recall:
for fuzzy sets $A$, $B$ on a reference set $X$, given by the corresponding membership functions $A(x)$ and $B(x)$:

$\bar{A}(x) = 1 - A(x)$ \quad – fuzzy complement

$(A \cap B)(x) = \min[A(x), B(x)]$ \quad – fuzzy intersection

$(A \cup B)(x) = \max[A(x), B(x)]$ \quad – fuzzy union

for all $x \in X$.

Properties of the standard operations

- They are generalizations of the corresponding (uniquely defined!) classical set operations.
- They satisfy the cutworthy and strong cutworthy properties. They are the only ones that do.
- The standard fuzzy intersection of two sets contains (is bigger than) all other fuzzy intersections of those sets.
- The standard fuzzy union of two sets is contained in (is smaller than) all other fuzzy unions of those sets.
- They inherently prevent the compound of errors of the operands.
Other generalizations of the set operations

Aggregation operators

Recall:
Aggregation operators are used to combine several fuzzy sets in order to produce a single fuzzy set.

**Associative** aggregation operations
- (general) fuzzy intersections - $t$-norms
- (general) fuzzy unions - $t$-conorms

**Non-associative** aggregation operations
- averaging operations - **idempotent** aggregation operations

Fuzzy complements

Axiomatic requirements

**Axiom c1.** $c(0) = 1$ and $c(1) = 0$. **boundary condition**

**Axiom c2.** For all $a, b \in [0, 1]$, if $a \leq b$, then $c(a) \geq c(b)$. **monotonicity**

$c_1$ and $c_2$ are called **axiomatic skeleton for fuzzy complements**

**Axiom c3.** $c$ is a **continuous** function.

**Axiom c4.** $c$ is **involutive**, i.e., $c(c(a)) = a$, for each $a \in [0, 1]$.

**Theorem**

*Let a function $c : [0, 1] \rightarrow [0, 1]$ satisfy **Axiom c2** and **Axiom c4**. Then $c$ satisfies Axioms **Axiom c1** and **Axiom c3** too. Moreover, the function $c$ is a bijection.*

Fuzzy complements

Definiton

- A fuzzy complement $cA$ of a fuzzy set $A$ is given by a function $c: [0, 1] \rightarrow [0, 1]$
- Function $c$ assigns a value to each membership value $A(x)$ of $x \in X$ to the fuzzy set $A$.
- A membership function of the set $cA$ is defined as $cA(x) = c(A(x))$.

Note:
- the value $cA(x)$ is interpreted not only as the degree to which $x \in X$ belongs to the fuzzy set $cA$, but also as the degree to which $x$ does not belong to the fuzzy set $A$;
- the value $cA(x)$ does not depend on $x$, but only on $A(x)$.

Nested structure of the basic classes of fuzzy complements

- All functions $c : [0, 1] \rightarrow [0, 1]$ 
  *IsNotC(a) = a* 
- All fuzzy complements (**Axiom c1** and **Axiom c2**) 
  $c(a) = \begin{cases} 
  1 & \text{for } a \leq t \\
  0 & \text{for } a > t 
  \end{cases}$
- All continuous fuzzy complements (**Axiom c1**- **Axiom c3**) 
  $c(a) = \frac{1}{2}(1 + \cos \pi a)$ 
  $c(\frac{1}{2}) = ?$
- All involutive fuzzy complements (**Axiom c1**- **Axiom c4**) 
  $c_\lambda(a) = \frac{1 - a}{1 + \lambda a}$, $\lambda > -1$ (Sugeno class) 
  $c_\omega(a) = (1 - a^\omega)^{\frac{1}{\omega}}$, $\omega > 0$ (Yager class) 
  $cA(x) = 1 - A(x)$ **Classical fuzzy complement**
Generators

Increasing generators

- **Increasing generator** is a strictly increasing continuous function \(g : [0, 1] \rightarrow R\), such that \(g(0) = 0\).
- **A pseudo-inverse** of increasing generator \(g\) is defined as:
  \[
g^{-1}(a) = \begin{cases} 
  0 & \text{for } a \in (-\infty, 0) \\
  g^{-1}(a) & \text{for } a \in [0, g(1)] \\
  1 & \text{for } a \in (g(1), \infty)
  \end{cases}
\]

- An example:
  \[
g(a) = e^p, \quad p > 0
\]
  \[
g^{-1}(a) = \begin{cases} 
  0 & \text{for } a \in (-\infty, 0) \\
  a^p & \text{for } a \in [0, 1] \\
  1 & \text{for } a \in (1, \infty)
  \end{cases}
\]

Decreasing generators

- **Decreasing generator** is a strictly decreasing continuous function \(f : [0, 1] \rightarrow R\), such that \(f(1) = 0\).
- **A pseudo-inverse** of increasing generator \(f\) is defined as:
  \[
f^{-1}(a) = \begin{cases} 
  1 & \text{for } a \in (-\infty, 0) \\
  f^{-1}(a) & \text{for } a \in [0, f(0)] \\
  0 & \text{for } a \in (f(0), \infty)
  \end{cases}
\]

- An example:
  \[
f(a) = 1 - a^p, \quad p > 0
\]
  \[
f^{-1}(a) = \begin{cases} 
  1 & \text{for } a \in (-\infty, 0) \\
  (1 - a)^p & \text{for } a \in [0, 1] \\
  0 & \text{for } a \in (1, \infty)
  \end{cases}
\]

Generating fuzzy complements

**Theorem** (First Characterization Theorem of Fuzzy Complements.)

Let \(c\) be a function from \([0, 1]\) to \([0, 1]\). Then \(c\) is an (involutive) fuzzy complement iff there exists an increasing generator \(g\) such that, for all \(a \in [0, 1]\)

\[c(a) = g^{-1}(g(1) - g(a)).\]

**Theorem** (Second Characterization Theorem of Fuzzy Complements.)

Let \(c\) be a function from \([0, 1]\) to \([0, 1]\). Then \(c\) is an (involutive) fuzzy complement iff there exists a decreasing generator \(f\) such that, for all \(a \in [0, 1]\)

\[c(a) = f^{-1}(f(0) - f(a)).\]
An intersection of two fuzzy sets $A$ and $B$ is given by a function of the form

$$i : [0, 1] \times [0, 1] \rightarrow [0, 1].$$

A value is assigned to a pair of membership values $A(x)$ and $B(x)$ of an element $x$ of the universal set $X$. It represents membership of $x$ to the intersection of $A$ and $B$:

$$(A \cap B)(x) = i(A(x), B(x)), \quad \text{for } x \in X.$$

Note:
- Intuitive requirements to be fulfilled by a function $i$ to qualify as an intersection of fuzzy sets are those of well known and extensively studied $t$-norms (triangular norms); the names fuzzy intersection and $t$-norm are therefore used interchangeably in the literature.
- The value $(A \cap B)(x)$ does not depend on $x$, but only on $A(x)$ and $B(x)$.

For all $a, b, d \in [0, 1],$

Ax $i_1$. $i(a, 1) = a$. boundary condition

Ax $i_2$. $b \leq d$ implies $i(a, b) \leq i(a, d)$. monotonicity

Ax $i_3$. $i(a, b) = i(b, a)$. commutativity

Ax $i_4$. $i(a, i(b, d)) = i(i(a, b), d)$. associativity

Axioms $i_1$ - $i_4$ are called axiomatic skeleton for fuzzy intersections.

If the sets are crisp, $i$ becomes the classical (crisp) intersection.

For all $a, b, d \in [0, 1],$

Ax $i_5$. $i$ is a continuous function. continuity

Ax $i_6$. $i(a, a) \leq a$. subidempotency

Ax $i_7$. $a_1 < a_2$ and $b_1 < b_2$ implies $i(a_1, b_1) < i(a_2, b_2)$. strict monotonicity

Note:
Subidempotency is a weaker requirement than idempotency, $i(a, a) = a$. A continuous subidempotent $t$-norm is called Archimedean $t$-norm.

The standard fuzzy intersection, $i(a, b) = \min[a, b]$, is the only idempotent $t$-norm.
Fuzzy intersections

Examples of t-norms frequently used

- **Drastic intersection**
  \[ i(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \]

- **Bounded difference**
  \[ i(a, b) = \max[0, a + b - 1] \]

- **Algebraic product**
  \[ i(a, b) = ab \]

- **Standard intersection**
  \[ i(a, b) = \min[a, b] \]

Fuzzy intersections

How to generate t-norms

Theorem

(Characterization Theorem of t-norms) Let \( i \) be a binary operation on the unit interval. Then, \( i \) is an Archimedean t-norm iff there exists a decreasing generator \( f \) such that

\[ i(a, b) = f^{-1}(f(a) + f(b)), \quad \text{for } a, b \in [0, 1]. \]

Example: A class of decreasing generators \( f_\omega(a) = (1 - a)^\omega, \quad \omega > 0 \)

generates a Yager class of t-norms

\[ i_\omega(a, b) = 1 - \min[1, ((1 - a)^\omega + (1 - b)^\omega)^{\frac{1}{\omega}}] \], \quad \omega > 0. \]

It can be proved that \( i_{\min}(a, b) \leq i_\omega(a, b) \leq \min[a, b] \).

Fuzzy unions

Definition

A **union** of two fuzzy sets \( A \) and \( B \) is given by a function of the form

\[ u : [0, 1] \times [0, 1] \rightarrow [0, 1]. \]

A value is assigned to a pair of membership values \( A(x) \) and \( B(x) \) of an element \( x \) of the universal set \( X \). It represents membership of \( x \) to the union of \( A \) and \( B \):

\[ (A \cup B)(x) = u(A(x), B(x)), \quad \text{for } x \in X. \]

Note:

- Intuitive requirements to be fulfilled by a function \( u \) to qualify as a union of fuzzy sets are those of well known and extensively studied t-conorms (triangular conorms); the names **fuzzy union** and **t-conorm** are therefore used interchangeably in the literature.
- The value \((A \cup B)(x)\) does not depend on \( x \), but only on \( A(x) \) and \( B(x) \).
Fuzzy Sets
Fuzzy techniques
Nataša Sladoje

Introduction
Types of set operations
Fuzzy complements
Fuzzy intersections
Fuzzy unions
Combinations of operations
Aggregation operations
An application: Fuzzy morphologies

Fuzzy unions
Axiomatic requirements

For all \(a, b, d \in [0, 1]\),

Ax u1. \(u(a, 0) = a\). boundary condition
Ax u2. \(b \leq d\) implies \(u(a, b) \leq u(a, d)\). monotonicity
Ax u3. \(u(a, b) = u(b, a)\). commutativity
Ax u4. \(u(a, u(b, d)) = u(u(a, b), d)\). associativity

Axioms u1 - u4 are called axiomatic skeleton for fuzzy unions. They differ from the axiomatic skeleton of fuzzy intersections only in boundary condition.

For crisp sets, \(u\) behaves like a classical (crisp) union.

Fuzzy unions
Examples of \(t\)-conorms frequently used

- Drastic union
  \[ u(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases} \]

- Bounded sum
  \[ u(a, b) = \min[1, a + b] \]

- Algebraic sum
  \[ u(a, b) = a + b - ab \]

- Standard intersection
  \[ u(a, b) = \max[a, b] \]

Fuzzy unions
Additional (optional) requirements

For all \(a, b, d \in [0, 1]\),

Ax u5. \(u\) is a continuous function. continuity
Ax u6. \(u(a, a) \geq a\). superidempotency
Ax u7. \(a_1 < a_2\) and \(b_1 < b_2\) implies \(u(a_1, b_1) < u(a_2, b_2)\). strict monotonicity

Note:
Requirements u5 - u7 are analogous to Axioms i5 - i7. Superidempotency is a weaker requirement than idempotency. A continuous superidempotent \(t\)-conorm is called Archimedean \(t\)-conorm.

The standard fuzzy union, \(u(a, b) = \max[a, b]\), is the only idempotent \(t\)-conorm.

Fuzzy unions
Properties

- \(\max[a, b] \leq a + b - ab \leq \min(1, a + b) \leq u_{\max}(a, b)\).
- For all \(a, b \in [0, 1]\), \(\max[a, b] \leq u(a, b) \leq u_{\max}(a, b)\).
Fuzzy unions
How to generate t-conoms

Theorem
(Characterization Theorem of t-conoms) Let \( u \) be a binary operation on the unit interval. Then, \( u \) is an Archimedean t-conorm if there exists an increasing generator \( g \) such that
\[
    u(a, b) = g^{-1}(g(a) + g(b)), \quad \text{for } a, b \in [0, 1].
\]

Example: A class of increasing generators \( f_\omega(a) = a^\omega, \ \omega > 0 \) generates a Yager class of t-conoms
\[
    u_\omega(a, b) = \min[1, (a^\omega + b^\omega)^{\frac{1}{\omega}}], \ \omega > 0.
\]

It can be proved that \( \max[a, b] \leq u_\omega(a, b) \leq \min[a, b] \).

Duality of fuzzy set operations
Examples of dual triples

Dual triples with respect to the standard fuzzy complement
\[
    \langle \min(a, b), \max(a, b), c_s \rangle
\]
\[
    \langle ab, a + b - ab, c_s \rangle
\]
\[
    \langle \max(0, a + b - 1), \min(1, a + b), c_s \rangle
\]
\[
    \langle \min_{\text{max}}(a, b), \max_{\text{max}}(a, b), c_s \rangle
\]

Combinations of set operations
De Morgan laws in classical set theory:
\[
    \overline{A \cap B} = A \cup \overline{B} \quad \text{and} \quad \overline{A \cup B} = A \cap \overline{B}.
\]

The union and intersection operation are dual with respect to the complement.

De Morgan laws for fuzzy sets:
\[
    c(i(A, B)) = u(c(A), c(B)) \quad \text{and} \quad c(u(A, B)) = i(c(A), c(B))
\]
for a t-norm \( i \), a t-conorm \( u \), and fuzzy complement \( c \).

Notation: \( \langle i, u, c \rangle \) denotes a dual triple.

Dual triples - Six theorems (1)

Theorem
The triples \( \langle \min, \max, c \rangle \) and \( \langle \min_{\text{max}}, \max_{\text{max}}, c \rangle \) are dual with respect to any fuzzy complement \( c \).

Theorem
Given a t-norm \( i \) and an involutive fuzzy complement \( c \), the binary operation \( u \) on \([0, 1]\), defined for all \( a, b \in [0, 1] \) by
\[
    u(a, b) = c(i(c(a), c(b)))
\]
is a t-conorm such that \( \langle i, u, c \rangle \) is a dual triple.

Theorem
Given a t-conorm \( u \) and an involutive fuzzy complement \( c \), the binary operation \( i \) on \([0, 1]\), defined for all \( a, b \in [0, 1] \) by
\[
    i(a, b) = c(u(c(a), c(b)))
\]
is a t-norm such that \( \langle i, u, c \rangle \) is a dual triple.
Theorem
Given an involutive fuzzy complement \( c \) and an increasing generator \( g \) of \( c \), the \( t \)-norm and the \( t \)-conorm generated by \( g \) are dual with respect to \( c \).

Theorem
Let \( (i, u, c) \) be a dual triple generated by an increasing generator \( g \) of the involutive fuzzy complement \( c \). Then the fuzzy operations \( i, u, c \) satisfy the law of excluded middle, and the law of contradiction.

Theorem
Let \( (i, u, c) \) be a dual triple that satisfies the law of excluded middle and the law of contradiction. Then \((i, u, c)\) does not satisfy the distributive laws.

Axiomatic requirements

**Ax h1** \( h(0, 0, \ldots, 0) = 0 \) and \( h(1, 1, \ldots, 1) = 1. \) boundary conditions

**Ax h2** For any \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\), such that \( a_i, b_i \in [0, 1] \) and \( a_i \leq b_i \) for \( i = 1, \ldots, n \),
\[
h(a_1, a_2, \ldots, a_n) \leq h(b_1, b_2, \ldots, b_n).
\]
\( h \) is monotonic increasing in all its arguments.

**Ax h3** \( h \) is continuous.

**Ax h4** \( h \) is a symmetric function in all its arguments; for any permutation \( p \) on \( \{1, 2, \ldots, n\} \)
\[
h(a_1, a_2, \ldots, a_n) = h(a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)}).
\]

**Ax h5** \( h \) is an idempotent function; for all \( a \in [0, 1] \)
\[
h(a, a, \ldots, a) = a.
\]

Aggregation operations

Aggregations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set.

**Definition**
Aggregation operation on \( n \) fuzzy sets \((n \geq 2)\) is a function \( h : [0, 1]^n \rightarrow [0, 1] \).

Applied to fuzzy sets \( A_1, A_2, \ldots, A_n \), function \( h \) produces an aggregate fuzzy set \( A \), by operating on membership grades to these sets for each \( x \in X \):
\[
A(x) = h(A_1(x), A_2(x), \ldots, A_n(x)).
\]

**Averaging operations**

- If an aggregation operator \( h \) is monotonic and idempotent (Ax h2 and Ax h5), then for all \((a_1, a_2, \ldots, a_n) \in [0, 1]^n \)
\[
\min(a_1, a_2, \ldots, a_n) \leq h(a_1, a_2, \ldots, a_n) \leq \max(a_1, a_2, \ldots, a_n).
\]
- All aggregation operators between the standard fuzzy intersection and the standard fuzzy union are idempotent.
- The only idempotent aggregation operators are those between standard fuzzy intersection and standard fuzzy union.

Idempotent aggregation operators are called averaging operations.
Averaging operations

**Generalized means:**

\[ h_\alpha(a_1, a_2, \ldots, a_n) = \left( a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha \right)^{\frac{1}{\alpha}}, \]

for \( \alpha \in \mathbb{R} \), and \( \alpha \neq 0 \), and for \( \alpha < 0 \) \( a_i \neq 0 \).

- **Geometric mean:** For \( \alpha \to 0 \),
  \[ \lim_{\alpha \to 0} h_\alpha(a_1, a_2, \ldots, a_n) = (a_1 \cdot a_2 \cdots a_n)^{\frac{1}{n}}; \]

- **Harmonic mean:** For \( \alpha = -1 \),
  \[ h_{-1}(a_1, a_2, \ldots, a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}; \]

- **Arithmetic mean:** For \( \alpha = 1 \),
  \[ h_1(a_1, a_2, \ldots, a_n) = \frac{1}{n}(a_1 + a_2 + \cdots + a_n). \]

Functions \( h_\alpha \) satisfy axioms **Ax h1 - Ax h5**.

Norm operations

Aggregation operations \( h \) on \([0, 1]^2\) which are

- monotonic
- commutative
- associative
- fulfil boundary conditions \( h(0, 0) = 0 \) and \( h(1, 1) = 1 \)

are called **norm operations**.

A class of norm operations

- contains \( t \)-norms and \( t \)-conorms, as special cases, which fulfill stronger boundary conditions;
- covers the whole range of aggregation operators, from \( l_{\min} \) to \( u_{\max} \);

Ordered Weighted Averaging Operations - OWA

For

- a given weighting vector \( w = (\omega_1, \omega_2, \ldots, \omega_n) \),
  \[ \omega_i \in [0, 1] \text{ for } i = 1, \ldots, n \]
  \[ \sum_{i=1}^{n} \omega_i = 1 \]
- a permutation \((b_1, b_2, \ldots, b_n)\) of a given vector \((a_1, a_2, \ldots, a_n)\)
  \[ b_i \geq b_j \quad \text{if} \quad i < j \]
  for any pair \(i, j \in \{1, 2, \ldots, n\}\)

an **OWA operation** associated with \( w \) is defined as

\[ h_w(a_1, a_2, \ldots, a_n) = \omega_1 b_1 + \omega_2 b_2 + \cdots + \omega_n b_n. \]

**Example:** For \( w = (0.3, 0.1, 0.2, 0.4) \),
\[ h_w(0.6, 0.9, 0.2, 0.7) = 0.3 \cdot 0.9 + 0.1 \cdot 0.7 + 0.2 \cdot 0.6 + 0.4 \cdot 0.2 = 0.54. \]

- **OWA operations** satisfy axioms **Ax h1 - Ax h5**
- **OWA operations** cover the whole range between \( \min \) and \( \max \) operations.

Norm operations

**Example:** \( \lambda \)-averages

For a given \( t \)-norm \( i \), and given \( t \)-conorm \( u \) for \( a, b \in [0, 1] \) and \( \lambda \in (0, 1) \)

\[ h_{(a,b)} = \left\{ \begin{array}{ll} \min[\lambda, u(a, b)] & \text{when } a, b \in [0, \lambda] \\ \max[\lambda, i(a, b)] & \text{when } a, b \in [\lambda, 1] \\ \lambda & \text{otherwise} \end{array} \right\} \]

is a parametrized class of norm operations which are neither \( t \)-norms, nor \( t \)-conorms.

These operations are called **\( \lambda \)-averages**.
Averaging operations
Do we need more than standard operations?

Making decisions in fuzzy environment:
Taking into account objectives and constraints expressed by fuzzy sets, make a new fuzzy set representing a decision.


1. “The board of directors tries to find the ‘optimal’ dividend to be paid to the shareholders. For financial reasons it should be attractive and for reasons of wage negotiations it should be modest.”
The decision about optimal dividend is based on the fuzzy set

\[ \mu_{\text{Optimal}}(x) = \min[\mu_{\text{Attractive}}(x), \mu_{\text{Modest}}(x)]. \]

2. “The teacher has to decide a student’s grade on a written test. The given task was supposed to be solved by Method1 or by Method2. The student provided two solutions, using both methods.”
The decision about his mark is based on a fuzzy set

\[ \mu_{\text{GoodSol}}(x) = \max[\mu_{\text{Good1stSol}}(x), \mu_{\text{Good2ndSol}}(x)]. \]

An example

- Two independent criteria, \( cr_1 \) and \( cr_2 \), are used to determine the quality of a product.
- For a number of products, the memberships to three fuzzy sets are experimentally determined:
  \[ \mu_{cr_1}, \mu_{cr_2}, \text{ and } \mu_{\text{Ideal}} \]
- Numbers: 60 people rating 24 products by assigning a value between 0 and 100, expressing memberships to:
  “FulfilledCr1”, “FulfilledCr2”, “IdealProduct”;
- Different theoretical aggregations (minimum, maximum, arithmetic mean, geometric mean) are computed and compared with the experimentally obtained “Ideal”.

Decision as a fuzzy set

Intersection: No positive compensation (trade-off) between the memberships of the fuzzy sets observed.
Union: Full compensation of lower degrees of membership by the maximal membership.

In reality of decision making, rarely either happens.

(non-verbal) “merging connectives” \( \rightarrow \) (language) connectives \{‘and’, ‘or’,...,\}.

Aggregation operations called compensatory and are needed to model fuzzy sets representing to, e.g., managerial decisions.

Theory vs. Experiments
Conclusions

- Geometric mean gives the best prediction of the empirical data, among the operations tested.
- Humans use other connectives than “and” or “or”; they need other aggregation operations than min and max.
- Even if the criteria are independent, and the corresponding fuzzy sets do not “interact”, the aggregation operation itself can “put them into interaction”.
- Every concrete decision set may require a specific aggregation operation.
- A general connective is required to involve some parameter corresponding to “grade of compensation”.
- The suggested operation is a weighted combination of non-compensatory ‘and’ and fully compensatory ‘or’ (in this work, they are interpreted as the product and the algebraic sum).

An Application: Fuzzy morphologies

Morphological operations

- Mathematical morphology is completely based on set theory. Fuzzification started in 1980s.
- Basic morphological operations are dilation and erosion. Many others can be derived from them.
- Dilation and erosion are, in crisp case, dual operations with respect to the complementation: $D(A) = c(E(cA))$.
- In crisp case, dilation and erosion fulfil a certain number of properties.

Aggregation by $\gamma$-operator

$\gamma$-operation, for $\gamma \in [0, 1]$, is defined as

$$\mu_{A \oplus B} = (1-\gamma)\mu_A \land \mu_B + \gamma\mu_A \lor \mu_B.$$  

The value $\gamma = 0.562$ is determined from the experimental data, for the chosen interpretation of intersection and union.

Dilation and erosion

Definitions

Definition

Dilation of a set $A$ by a structuring element $B$ is

$$D_B(A) = A \oplus B = \{ x \in X | \tau_x(B) \cap A \neq \emptyset \}.$$  

$\tau_a(S) = \{ y | y = a + s, s \in S \}$

$\hat{S} = \{ y | y = -s, s \in S \}$.

Definition

Erosion of a set $A$ by a structuring element $B$ is

$$E_B(A) = A \ominus B = \{ a \in X | \tau_a(B) \subseteq A \}.$$
Dilation and erosion
An example

Dilation and erosion of a shape by a circular structuring element.

How to construct fuzzy mathematical morphology
\(\alpha\)-cut decomposition

For fuzzy sets \(\mu(x)\) and \(\nu(x)\) it holds that

\[
[\mu(x) \oplus \nu(x)]_{\alpha}(x) = \mu_{\alpha}(x) \oplus \nu_{\alpha}(x).
\]

Steps to take:
- \(\alpha\)-cut decomposition of both the set and the structuring element;
- performance of (crisp) morphological operations;
- reconstruction of a resulting fuzzy set from its obtained \(\alpha\)-cuts.

How to construct fuzzy mathematical morphology
\(\alpha\)-cut decomposition

It is shown that this approach leads to definitions:

\[
D_\alpha(\mu)(x) = \int_0^1 \sup_{y \in (\nu_{\alpha})_x} \mu(y)d\alpha
\]

\[
E_\alpha(\mu)(x) = \int_0^1 \inf_{y \in (\nu_{\alpha})_x} \mu(y)d\alpha
\]

if a reconstruction of a fuzzy set is performed by using

\[
\mu(x) = \int_0^1 \mu_{\alpha}(x)d\alpha
\]
How to construct fuzzy mathematical morphology

**α-cut decomposition**

... or

\[
D_\nu(\mu)(x) = \sup_{y \in X} [\nu(y - x)\mu(y)] \\
E_\nu(\mu)(x) = \inf_{y \in X} [\mu(y)\nu(y - x) + 1 - \nu(y - x)]
\]

if a reconstruction of a fuzzy set is performed by using

\[
\mu(x) = \sup_{\alpha \in [0,1]} [\alpha\mu_\alpha(x)].
\]

Most general definitions of fuzzy dilation and fuzzy erosion obtained by fuzzification of set operations are:

\[
D_\nu(\mu)(x) = \sup_{y \in X} i(\nu(y - x), \mu(y)) \\
E_\nu(\mu)(x) = \inf_{y \in X} u(c(\mu(y), \nu(y - x))),
\]

where \(i\) is any \(t\)-norm and \(u\) is its dual \(t\)-conorm with respect to a fuzzy complement \(c\).

Perform "fuzzification" of all (set) operations and relations used for definitions of morphological operations.

- Replace unions and intersections by \(t\)-conorms and \(t\)-norms,
- Use fuzzy complementation
- Replace subsethood relation by inclusion indicators
- Use duality of morphological operations
- ...

How to construct fuzzy mathematical morphology

Fuzzification of set operations

A fuzzy erosion can be defined as

\[
E_\nu(\mu)(x) = I(\nu + x, \mu)
\]

where \(I(\nu, \mu)\) is an inclusion indicator, a function that fulfils a certain set of axioms. (Several exist in the literature.)

Then, dilation is defined by duality principle.

Alternatively, a fuzzy dilation is defined as

\[
D_\nu(\mu)(x) = O(\nu + x, \mu)
\]

where \(O\) is an overlapping indicator.
A task on fuzzy morphology

Use the paper:

- Design fuzzy morphological operations. Use $\alpha$-cut decomposition principle, and fuzzification of set operation by $t$-norms and $t$-conorms.
- Implement at least three different morphological dilations and corresponding erosions; use 2-3 different structuring elements; take simple (one-dimensional) examples; look at the examples in the paper.
- Comment the results. Observe visual differences and think of their effect in possible applications. Study the properties of the morphological operations, using your obtained results and the paper.
- Make a short summary.