Fuzzy Sets and Fuzzy Techniques
Lecture 6 – Distances on and between fuzzy sets

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A note on the estimation of perimeter

The perimeter of a fuzzy set $A$

$$\text{perim}(A) = \int_0^1 \text{perim}(\mu_A) \, d\alpha$$

The perimeter of a fuzzy step set $S$ given by a piecewise constant membership function $\mu_S$, is defined as

$$\text{perim}(S) = \sum_{i,j,k \mid i \neq j} |\mu_S_i - \mu_S_j| \cdot |A_{ijk}|,$$

where $A_{ijk}$ is the $k^{th}$ arc along which bounded regions $S_i$ and $S_j$, defined by (constant-valued) membership functions $\mu_S_i$ and $\mu_S_j$, meet.

Note: requires estimation of the arc-lengths of the crisp sets.

Topics of today

- Distances I
  - Point to point distances
  - Point to set distances
  - Set to set distances

Pal and Rosenfeld uses the most simple estimate, that is directly counting pixel edges between foreground and background.
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Outline
A note on the estimation of perimeter

Matlab

function p=perim_2d(I)
a=abs(diff(I,[],1)); %horizontal edges
b=abs(diff(I,[],2)); %vertical edges
p=sum(a(:))+sum(b(:));

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A note on the estimation of perimeter

Distances

Pal and Rosenfeld uses the most simple estimate, that is directly counting pixel edges between foreground and background.

A more precise method is described in:

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Binary case

For a binary configuration there are $2^4 = 16$ possible configurations of pixels forming a Marching Square. Grouping symmetry and complementary cases results in 4 different configurations. The length of the border, corresponding to these configurations, is equal to

- 0, when all 4 pixels are set or when none of them is set;
- $\frac{b}{2}$, when one of the 4 pixels is set or when one of them is not set;
- $a$, when two edge-neighbours are set;
- $b$, when two vertex-neighbours are set,

where $a$ and $b$ are the weights assigned to an isothetic and a diagonal step between two neighbouring pixels, respectively. We use $a_{MSE_{n \to \infty}} \approx 0.948$ and $b_{MSE_{n \to \infty}} \approx 1.343$ to minimize the expected mean square error ($MSE$) for measurement of the length of long line segments ($n \to \infty$) and to give an unbiased estimate. (Note that the weights $a = 1$ and $b = \sqrt{2}$ are not optimal.)

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Fuzzy case

The perimeter contribution $per$ of a fuzzy $2 \times 2$ configuration $c_i$ to the perimeter of a fuzzy object is

$$per(c_i) = \sum_{j=1}^{4} d_j \cdot b_j , \text{ for } n = 2 ,$$

(1)

where for $j = 1, 2, 3$, $d_j = \mu_S(x_{j+1}) - \mu_S(x_j)$, $\mu_S(x_j)$ is the membership value of the pixel $x_j$ of $c_i$, and the memberships within the configuration are sorted in non-decreasing order, i.e., $\mu_S(x_j) \leq \mu_S(x_{j+1})$, while $b_j$ is the estimated length of the line corresponding to the binary $2 \times 2$ configuration obtained as an $\alpha$-cut of a fuzzy configuration $c_i$ at level $\alpha = \mu_S(x_{j+1})$.

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The perimeter of a fuzzy object is obtained as the sum of the contributions of all fuzzy $2 \times 2$ configurations contained in the fuzzy segmented image:

$$\text{Per} = \sum_i \text{per}(c_i).$$

A good overview:


**Figure**: Calculation of the contribution to perimeter of a fuzzy Marching Square.
## Definitions

**Definition**

A **metric** is a positive function $d$ such that

1. $d(x, x) = 0$ (reflexivity)
2. $d(x, y) = 0 \Rightarrow x = y$ (separability)
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality)

- If we drop requirement 2, we have a **pseudometric**
- If we drop requirement 4, we have a **semimetric**

A **semi-pseudometric** thus satisfies only 1 and 3.