Fuzzy Sets and Fuzzy Techniques

Lecture 1 – Introduction

Joakim Lindblad
joakim@cb.uu.se

Centre for Image Analysis
Uppsala University

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Topics of today

- What is a fuzzy set?
- What can fuzzy sets and fuzzy techniques be used for?
- About the course
  - What will we learn in this course?
  - Schedule, exercises, project, exam...
- Starting with Chapter 1...
  - Fuzzy sets: Basic types, notions and concepts

What is a fuzzy set?

Btw., what is a set? “... to be an element...”

A **set** is a collection of its **members**.

The notion of **fuzzy sets** is an extension of the most fundamental property of sets.

Fuzzy sets allows a grading of **to what extent** an **element** of a set **belongs** to that specific set.

What is a fuzzy set?

A small example

Let us observe a (crisp) reference set (our universe)

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Let us form:
The (crisp) subset C of X, $C = \{x \mid 3 < x < 8\}$

$$C = \{4, 5, 6, 7\}$$

(Easy! "Yes, or no" ...)

The set F of **big** numbers in X

$$F = \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$$

(Yes or no? ... More like graded ...)

Why Fuzzy?

Precision is not truth.

- Henri Matisse

So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.

- Albert Einstein

As complexity rises, precise statements lose meaning and meaningful statements lose precision.

- Lotfi Zadeh

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What is a fuzzy set?

Fuzzy is not just another name for probability.

The number 10 is not **probably** big!
...and number 2 is not **probably not** big.

Uncertainty is a consequence of
non-sharp boundaries between the notions/objects,
and not caused by lack of information.

Statistical models deal with random events and outcomes; fuzzy models attempt to capture and quantify nonrandom imprecision.

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What is a fuzzy set?

Randomness vs. Fuzziness

Randomness refers to an event that may or may not occur.
Randomness: frequency of car accidents.

Fuzziness refers to the boundary of a set that is not precise.
Fuzziness: seriousness of a car accident.

Prof. George J. Klir

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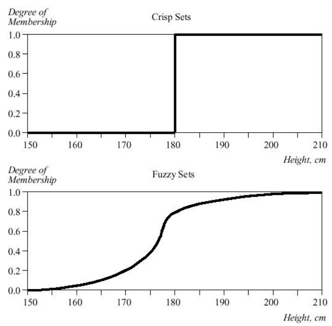
An example

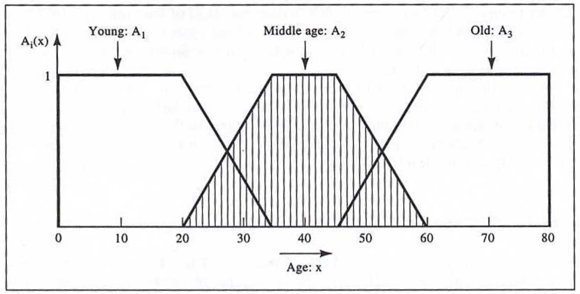
A fuzzy set of **tall men**

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00



Another example

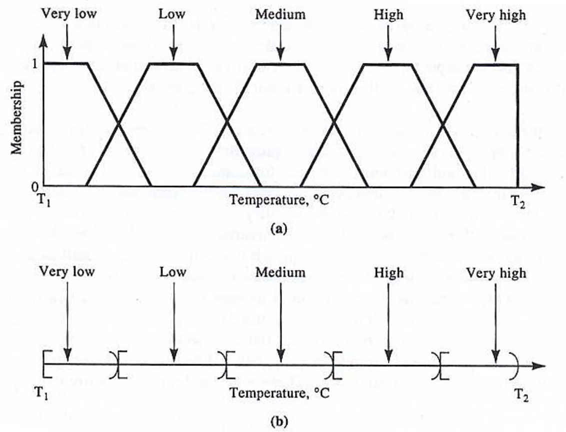
Age groups



Membership functions representing the concepts of a young, middle-aged, and old person.

Yet another example

Temperature



Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

Computing with Words

- Lexical Imprecision
- Linguistic Variable

Computing with Words

Medical Diagnosis

Medicine is one field in which the applicability of fuzzy set theory (FST) was recognized quite early (mid-1970s).

Diagnosis of disease has frequently been the focus of application of FST.

The process of classifying different sets of symptoms under a single name and determining appropriate therapeutic actions becomes increasingly difficult.

Computing with Words

Linguistic descriptions

The best and most useful descriptions of diseases entities often use **linguistic terms** that are irreducibly vague.

Example: **Hepatitis**

"Total proteins are **usually normal**, *albumin is* **decreased**, *alpha-globulin are* **slightly decreased**, *beta-globulins are* **slightly decreased**, *and gamma-globulins are* **increased***."*

The linguistic terms printed in **blue** color are inherently vague.

Computing with Words

Computing with Words (CW) is a methodology in which words are used in place of numbers for computing and reasoning.

CW is a necessity when the available information is too imprecise to justify the use of numbers.

By allowing a certain amount of tolerance for imprecision CW can be used to achieve tractability, robustness, low solution cost, and better connection with reality.

Fuzzy vs. probability

Fuzzy sets theory **complements** probability theory

Patients suffering from hepatitis show in 60% of all cases high fever, in 45% of all cases a yellowish colored skin, and in 30% of all cases nausea.

Lotfi A. Zadeh

The Father of Fuzzy

Lotfi Asker Zadeh

From Wikipedia, the free encyclopedia
(Redirected from Lotfi Zadeh)

Lotfi A. Zadeh in 2004

Lotfi Asker Zadeh (born February 4, 1921) is a mathematician and computer scientist, and a professor of computer science at the University of California, Berkeley.

He was born in Baku, Azerbaijan as **Lotfi Aliaskerzadeh** (or **Askar Zadeh**), to a Russian Jewish mother and an Azerbaijani father, grew up in Iran, studied at Alborz High School and Tehran University, and moved to the United States in 1944. He has taught at UC Berkeley since 1959. He published his seminal work on fuzzy set in 1965 in which he detailed the mathematics of fuzzy set theory. In 1973 he proposed his theory of fuzzy logic.

Lotfi Zadeh is noted to, "quick(ly) shrug off nationalism, insisting there are much deeper issues in life", where he himself is quoted stating, "The question really isn't whether I'm American, Russian, Iranian, Azerbaijani, or anything else, I've been shaped by all these people and cultures and I feel quite comfortable among all of them." [1]

Lotfi A. Zadeh

The Father of Fuzzy

Aristotle introduced the laws of thought which consisted of three fundamental laws:

- principle of identity
- law of the excluded middle
- law of contradiction

The law of the excluded middle states that for all propositions p, either p or $\neg p$ must be true, there being no middle true proposition between them. In other words, p cannot be both p and not p.

Plato laid the foundation of what is now known as fuzzy logic indicating that there was a third region beyond true and false.

It was **Jan Łukasiewicz** (in 1910) who first proposed a systematic alternative to the bi-valued logic of Aristotle and described the **3-valued logic**, with the third value being **Possible**.

Lotfi Zadeh, in his theory of fuzzy logic, proposed the making of the membership function operate over the range of real numbers [0,1]. He proposed new operations for the calculus of logic and showed that fuzzy logic was a generalization of classical logic.

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Lotfi A. Zadeh

Things to read

- L. A. Zadeh, Fuzzy sets. Information and Control, Vol. 8, pp. 338-353. (1965).
  `http://www-bisc.cs.berkeley.edu/zadeh/papers/Fuzzy%20Sets-1965.pdf`
- L. A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Transactions on Systems, Man and Cybernetics SMC-3, 28-44, 1973.

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How Big is "Fuzzy"?

Who knows? Zadeh is too busy pushing forward to keep up with how far the field has expanded. His office in the newly constructed Computer Science Building at Berkeley is stacked floor to ceiling with reprints of articles related to Fuzzy. He believes that people are studying this field in every country which offers advanced education. **Twelve journals** are now published which include the word "Fuzzy" in their title. An estimated **15,000 articles** have been published, although it's hard to be exact as some appear in obscure journals in remote parts of the world. An estimated 3,000 patents have been applied for and 1,000 granted. The Japanese, with 2,000 scientists involved in Fuzzy Logic, have been very quick to incorporate Fuzzy Logic in the design of consumer products, such as household appliances and electronic equipment and one company, Mitsushita (which sells under the name of Panasonic and Quasar) acknowledged that in 1991-1992 alone, they had sold more than **1 billion dollars** worth of equipment that used Fuzzy Logic. The concept is so popular there that the English word has entered the Japanese language, though the Japanese pronounce it more like "fudgy" than "fuzzy".

Short Biographical Sketch,

Azerbaijan International **1994**, by Betty Blair

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Not without controversy

L. Zadeh Fuzzy Logic 40 Years Later

What can it be used for?

A lot! :-)

About the course

Fuzzy Sets and Fuzzy Techniques

`http://www.cb.uu.se/~joakim/course/fuzzy`

- 15 lectures
- 2 computer exercises
- 1 small project work + presentation
- Written exam

What will we learn in this course?

Fuzzy Sets and Fuzzy Techniques

- The basics of fuzzy sets
  - How to define fuzzy sets
  - How to perform operations on fuzzy sets
  - How to extend crisp concepts to fuzzy ones
  - How to extract information from fuzzy sets
- The very basics of fuzzy logic and fuzzy reasoning
- We will look at some applications of fuzzy in
  - Image processing
  - Control systems
  - Machine intelligence / expert systems

Schedule

`http://www.cb.uu.se/~joakim/course/fuzzy/schedule.html`

Teachers

- Joakim Lindblad
- Nataša Sladoje (3 lectures)
- Laszlo Nyul (1 lecture)

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Computer exercises

- Something simple just to make sure that you are following

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Project work

Apply fuzzy in your own work

- Groups of two
- Compare with traditional (crisp)
- 15 min. presentation (8th of March)

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Exam

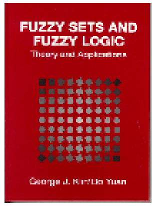
CBA 20th of March

- Will not be too hairy... but not all to easy either.

Course Literature

The book

http://www.cb.uu.se/~joakim/course/fuzzy/literature.html



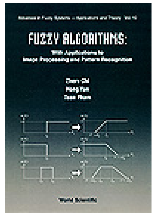
Fuzzy Sets and Fuzzy Logic: Theory and Applications

- Covers more than the course
- Emphasis on theory
- \+ Good and reliable
- – Hard to find
- – Expensive

Course Literature

Additional literature

http://www.cb.uu.se/~joakim/course/fuzzy/literature.html



Fuzzy Algorithms: With Applications to Image Processing and Pattern Recognition

- Will be used toward the end of the course (applications)

Chapter one...

From Ordinary (Crisp) Sets to Fuzzy Sets

A Grand Paradigm Shift

(Crisp) Set Theory

At the beginning of his work Beiträge zur Begründung der transfiniten Mengenlehre, **Georg Cantor**, the principal creator of set theory, made the following definition of a set:

"By a set we understand any collection M of definite, distinct objects m of our perception or of our thought (which will be called the ***elements*** *of M) into a whole."*

The objects of a set are also called its **members**. The elements of a set can be **anything**: numbers, people, letters of the alphabet, other sets, and so on.

1.2 Crisp sets: An overview

- Sets: $\mathbb{Z}$, $\mathbb{R}$, intervals, ordered pairs
- Notation: member, $\in$, $\notin$
- List, rule, characteristic fun.
- Set of sets = **family of sets**
- Subset ($\subseteq$), equality ($=$), inequality ($\neq$), proper subset ($\subset$)
- Power set $\mathcal{P}$ (higher orders)
- Cardinality ($|\cdot|$)
- Relative complement
- Universal set
- (Absolute) complement (is involutive)
- Union, intersection

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1.2 Crisp sets: An overview

- Table 1.1: Properties
- Principle of duality ($\emptyset \leftrightarrow X$ and $\cup \leftrightarrow \cap$)
- Partial ordering of the power set by set inclusion (forms a Lattice)
- Disjoint sets
- Partition (consisting of blocks), refinement
- A nested family of sets
- Cartesian product (all ordered pairs), n-tuples
- Subsets of Cartesian products are called relations
- Countable (finite and infinite) and uncountable sets
- Sets on $\mathbb{R}^n$
- Convex sets
- Upper/lower bound (set on $\mathbb{R}$) and sup and inf.

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Fuzzy sets

A fuzzy set of a reference set is a set of ordered pairs

$$F = \{\langle x, \mu_F(x)\rangle \mid x \in X\},$$

where $\mu_F : X \to [0,1]$.

Where there is no risk for confusion, we use the same symbol for the fuzzy set, as for its membership function.

Thus

$$F = \{\langle x, F(x)\rangle \mid x \in X\},$$

where $F : X \to [0,1]$.

To define a fuzzy set $\Leftrightarrow$ To define a membership function

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Fuzzy sets

Continuous (analog) fuzzy sets

$$A : X \to [0,1]$$

Discrete fuzzy sets

$$A : \{x_1, x_2, x_3, ..., x_s\} \to [0,1]$$

Digital fuzzy sets

If a discrete-universal membership function can take only a finite number $n \geq 2$ of distinct values, then we call this fuzzy set a digital fuzzy set.

$$A : \{x_1, x_2, x_3, ..., x_s\} \to \{0, \tfrac{1}{n-1}, \tfrac{2}{n-1}, \tfrac{3}{n-1}, ..., \tfrac{n-2}{n-1}, 1\}$$

Fuzzy sets

Universal set
X - is the universe of discourse, or universal set, which contains all the possible elements of concern in each particular context of applications.

Membership function (compare Characteristic function)
The membership function M maps each element of X to a membership grade (or membership value) between 0 and 1.

A fuzzy set M, in the universal set can be presented by:

- list form,
- rule form,
- membership function form.

Fuzzy sets

List form

$$M = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 3,0.9\rangle, \langle 4,0.7\rangle, \langle 5,0.3\rangle, ...\},$$

Note: The list form can be used only for finite sets.

Rule form

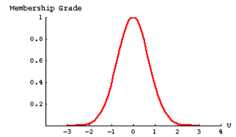
$$M = \{x \in X \mid x \text{ meets some conditions}\},$$

where the symbol | denotes the phrase "such that".

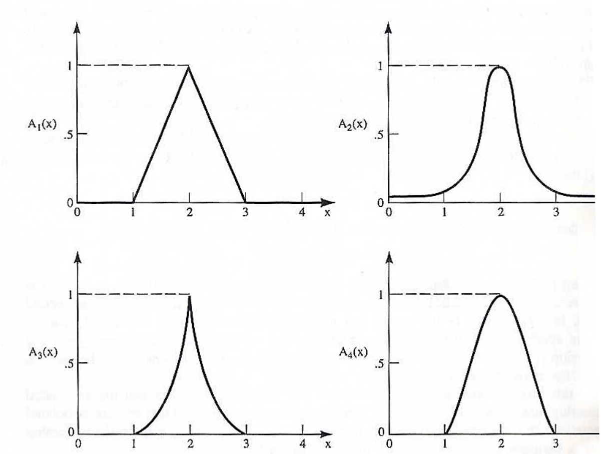
Membership form
Let M be a fuzzy set named "numbers closed to zero"

$$M(x) = e^{-x^2} \text{ for } x \in [-3, 3]$$

$$M(0) = 1, M(2) = e^{-4}, M(-2) = e^{-4}$$



Fuzzy sets



Examples of membership functions that may be used in different contexts for characterizing fuzzy sets of real numbers close to 2.

Fuzzy sets of different types

The membership function may be vague in itself.

Interval-valued fuzzy sets

$$A : X \to \mathcal{E}([0,1])$$

Fuzzy sets of **type 2**

$$A : X \to \mathcal{F}([0,1])$$

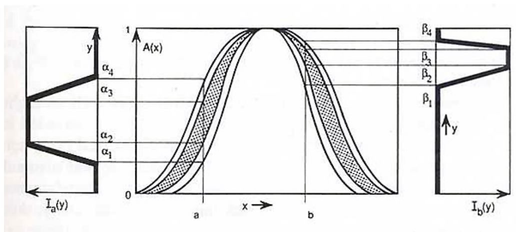


Illustration of the concept of a fuzzy set of type 2.

Types 3, 4, etc...

L-fuzzy sets – L is any **partially ordered** set

$$A : X \to L$$

Fuzzy sets of different levels

Also the **domain** of the membership function may be fuzzy.

Fuzzy sets defined so that the elements of the universal set are themselves fuzzy sets are called **level 2 fuzzy sets**.

$$A : \mathcal{F}(X) \rightarrow [0, 1]$$

Using a universal set containing level 2 fuzzy sets, we similarly get level 3 fuzzy sets, etc.

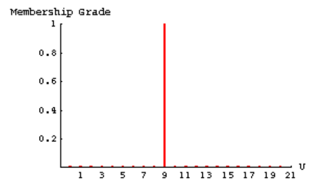
We will, however, stick to ordinary fuzzy sets, of type 1 and level 1.

Basic concepts and terminology

The **support** of a fuzzy set A in the universal set X is a crisp set that contains all the elements of X that have nonzero membership values in A, that is,

$$\text{supp}(A) = \{x \in X \mid A(x) > 0\}$$

A fuzzy **singleton** is a fuzzy set whose support is a single point in X.



Basic concepts and terminology

A **crossover point** of a fuzzy set is a point in X whose membership value to A is equal to 0.5.

The **height**, $h(A)$ of a fuzzy set A is the largest membership value attained by any point. If the height of a fuzzy set is **equal to one**, it is called a **normal** fuzzy set, otherwise it is **subnormal**.

Basic concepts and terminology

An **α-cut** of a fuzzy set A is a **crisp set** ${}^{\alpha}A$ that contains all the elements in X that have membership value in A greater than or equal to α.

$${}^{\alpha}A = \{x \mid A(x) \geq \alpha\}$$

A **strong α-cut** of a fuzzy set A is a crisp set ${}^{\alpha+}A$ that contains all the elements in X that have membership value in A **strictly** greater than α.

$${}^{\alpha+}A = \{x \mid A(x) > \alpha\}$$

Basic concepts and terminology

We observe that the strong α-cut ${}^{0+}A$ is equivalent to the support supp(A).

The 1-cut ${}^{1}A$ is often called the **core** of A.
Note! Sometimes the highest non-empty α-cut ${}^{h(A)}A$ is called the core of A. (in the case of subnormal fuzzy sets, this is different).

The word **kernel** is also used for both of the above definitions. (Total confusion!)

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Basic concepts and terminology

The ordering of the values of α in $[0,1]$ is **inversely** preserved by set inclusion of the corresponding α-cuts as well as strong α-cuts. That is, for any fuzzy set A and $\alpha_1 < \alpha_2$ it holds that ${}^{\alpha_2}A \subseteq {}^{\alpha_1}A$.

All α-cuts and all strong α-cuts for two distinct families of **nested** crisp sets.

The set of all levels $\alpha \in [0,1]$ that represent distinct α-cuts of a given fuzzy set A is called a **level set** of A.

$$\Lambda(A) = \{\alpha \mid A(x) = \alpha \text{ for some } x \in X\}.$$

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Basic concepts and terminology

A fuzzy set A defined on $\mathbb{R}^n$ is **convex** iff

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\left(A(x_1), A(x_2)\right),$$

for all $\lambda \in [0,1], x_1, x_2 \in \mathbb{R}^n$ and all $\alpha \in [0,1]$.

Or, equivalently, A is **convex** if and only if all its α-cuts ${}^{\alpha}A$, for any α in the interval $\alpha \in (0,1]$, are convex sets.

Any property that is generalized from classical set theory into the domain of fuzzy set theory by requiring that it holds in all α-cuts in the classical sense is called a **cutworthy** property.

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1.3 Fuzzy sets: Basic types

- Characteristic function $\rightarrow$ membership function
- Most common: universal set $X \rightarrow [0,1]$
- μ or not μ
- Membership functions of different shapes
- Fuzzy power set $\mathcal{F}$
- Fuzzy sets of different types (codomain) (ordinary (type 1), interval valued, type 2..., L-fuzzy sets)
- Different levels of fuzzy sets (domain) - fuzzy universal set

1.4 Fuzzy sets: Basic concepts

- alpha-cut and strong alpha-cut
- The level set Λ (subset range ([0,1]))
- Subset ordering of alpha cuts (inverse to alpha) -¿ nested crisp sets
- New way to define FS, set of alpha cuts
- Support, core, height, normal & subnormal
- Convexity (on $\mathbb{R}^n$) (not convex function!) [proof]
- Cutworthy and strong cutworthy properties (holds for all alpha cuts)

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1.4 Fuzzy sets: Basic concepts

- Standard fuzzy set operations
  - Complement, equilibrium points
  - Union & intersection
  - Lattice : De Morgan lattice/algebra - No law of contr. and excl. middle
  - Set inclusion (and equality)
  - Scalar cardinality (sigma count!)
  - Degree of subsethood
- Notation with slash: $A = A(x_1)/x_1 + A(x_2)/x_2 = \sum \ldots$ or $\int \ldots$ for continous sets
- Geometric interpretation (prob. distr. sum to 1)