What is a fuzzy set?

Btw., what is a set?

"... to be an element..."

A set is a collection of its members.

The notion of fuzzy sets is an extension of the most fundamental property of sets.

Fuzzy sets allows a grading of to what extent an element of a set belongs to that specific set.

What is a fuzzy set?

Let us observe a (crisp) reference set (our universe)

\[ X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \]

Let us form:

The (crisp) subset \( C \) of \( X \),

\[ C = \{ x \mid 3 < x < 8 \} \]

\[ C = \{4, 5, 6, 7\} \]

(Easy! "Yes, or no" ...)

The set \( F \) of big numbers in \( X \)

\[ F = \{10, 9, 8, 7, 6, 5, 4, 3, 2\} \]

(Yes or no? ... More like graded ...)
Why Fuzzy?

Precision is not truth.

- Henri Matisse

So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.

- Albert Einstein

As complexity rises, precise statements lose meaning and meaningful statements lose precision.

- Lotfi Zadeh

Fuzzy is not just another name for probability.

The number 10 is not probably big!

...and number 2 is probably not big.

Uncertainty is a consequence of non-sharp boundaries between the notions/objects, and not caused by lack of information.

Statistical models deal with random events and outcomes; fuzzy models attempt to capture and quantify nonrandom imprecision.

Randomness refers to an event that may or may not occur.

**Randomness**: frequency of car accidents.

Fuzziness refers to the boundary of a set that is not precise.

**Fuzziness**: seriousness of a car accident.

Prof. George J. Klir
Another example

Age groups

Yet another example

Temperature

Computing with Words

• Lexical Imprecision
• Linguistic Variable

Medical Diagnosis

Medicine is one field in which the applicability of fuzzy set theory (FST) was recognized quite early (mid-1970s). Diagnosis of disease has frequently been the focus of application of FST. The process of classifying different sets of symptoms under a single name and determining appropriate therapeutic actions becomes increasingly difficult.
The best and most useful descriptions of diseases entities often use linguistic terms that are irreducibly vague.

Example: Hepatitis

"Total proteins are usually normal, albumin is decreased, alpha-globulin are slightly decreased, beta-globulins are slightly decreased, and gamma-globulins are increased."

The linguistic terms printed in blue color are inherently vague.

Patients suffering from hepatitis show in 60% of all cases high fever, in 45% of all cases a yellowish colored skin, and in 30% of all cases nausea.
Aristotle introduced the laws of thought which consisted of three fundamental laws:

- principle of identity
- law of the excluded middle
- law of contradiction

The law of the excluded middle states that for all propositions \( p \), either \( p \) or \( \neg p \) must be true, there being no middle true proposition between them. In other words, \( p \) cannot be both \( p \) and not \( p \).

Plato laid the foundation of what is now known as fuzzy logic indicating that there was a third region beyond true and false.

It was Jan Łukasiewicz (in 1910) who first proposed a systematic alternative to the bi-valued logic of Aristotle and described the 3-valued logic, with the third value being Possible.

Lotfi Zadeh, in his theory of fuzzy logic, proposed the making of the membership function operate over the range of real numbers \([0,1]\). He proposed new operations for the calculus of logic and showed that fuzzy logic was a generalization of classical logic.

Lotfi A. Zadeh, The Father of Fuzzy

Lotfi A. Zadeh

How Big is "Fuzzy"?

Who knows? Zadeh is too busy pushing forward to keep up with how far the field has expanded. His office in the newly constructed Computer Science Building at Berkeley is stacked floor to ceiling with reprints of articles related to Fuzzy. He believes that people are studying this field in every country which offers advanced education. Twelve journals are now published which include the word “Fuzzy” in their title. An estimated 15,000 articles have been published, although it’s hard to be exact as some appear in obscure journals in remote parts of the world. An estimated 3,000 patents have been applied for and 1,000 granted. The Japanese, with 2,000 scientists involved in Fuzzy Logic, have been very quick to incorporate Fuzzy Logic in the design of consumer products, such as household appliances and electronic equipment and one company, Mitsushita (which sells under the name of Panasonic and Quasar) acknowledged that in 1991-1992 alone, they had sold more than 1 billion dollars worth of equipment that used Fuzzy Logic. The concept is so popular there that the English word has entered the Japanese language, though the Japanese pronounce it more like “fudgy” than “fuzzy”.

Short Biographical Sketch, Azerbaijan International 1994, by Betty Blair

Lotfi A. Zadeh

Things to read

What can it be used for?

A lot! :-)

About the course

http://www.cb.uu.se/~joakim/course/fuzzy

- 15 lectures
- 2 computer exercises
- 1 small project work + presentation
- Written exam

What will we learn in this course?

The basics of fuzzy sets
- How to define fuzzy sets
- How to perform operations on fuzzy sets
- How to extend crisp concepts to fuzzy ones
- How to extract information from fuzzy sets

The very basics of fuzzy logic and fuzzy reasoning
- Image processing
- Control systems
- Machine intelligence / expert systems

Schedule

http://www.cb.uu.se/~joakim/course/fuzzy/schedule.html
Teachers

- Joakim Lindblad
- Nataša Sladoje (3 lectures)
- Laszlo Nyul (1 lecture)

Computer exercises

- Something simple just to make sure that you are following

Project work

Apply fuzzy in your own work

- Groups of two
- Compare with traditional (crisp)
- 15 min. presentation (8th of March)

Exam

CBA 20th of March

- Will not be too hairy... but not all to easy either.
Fuzzy Sets and Fuzzy Logic: Theory and Applications
- Covers more than the course
- Emphasis on theory
- Good and reliable
  - Hard to find
  - Expensive

Fuzzy Algorithms: With Applications to Image Processing and Pattern Recognition
- Will be used toward the end of the course (applications)

(Crisp) Set Theory

At the beginning of his work Beiträge zur Begründung der transfiniten Mengenlehre, Georg Cantor, the principal creator of set theory, made the following definition of a set:

"By a set we understand any collection \( M \) of definite, distinct objects \( m \) of our perception or of our thought (which will be called the elements of \( M \)) into a whole."

The objects of a set are also called its members. The elements of a set can be anything: numbers, people, letters of the alphabet, other sets, and so on.
1.2 Crisp sets: An overview

- Sets: \( \mathbb{Z}, \mathbb{R}, \) intervals, ordered pairs
- Notation: member, \( \in, \notin \)
- List, rule, characteristic fun.
- Set of sets = family of sets
- Subset (\( \subseteq \)), equality (=), inequality (\( \neq \)), proper subset (\( \subset \))
- Power set \( \mathcal{P} \) (higher orders)
- Cardinality (\( |\cdot| \))
- Relative complement
- Universal set
- (Absolute) complement (is involutive)
- Union, intersection

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Fuzzy sets

A fuzzy set of a reference set is a set of ordered pairs

\[
F = \{(x, \mu_F(x)) \mid x \in X\},
\]

where \( \mu_F : X \rightarrow [0, 1] \).

Where there is no risk for confusion, we use the same symbol for the fuzzy set, as for its membership function.

Thus

\[
F = \{(x, F(x)) \mid x \in X\},
\]

where \( F : X \rightarrow [0, 1] \).

To define a fuzzy set \( \iff \) To define a membership function

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Continuous (analog) fuzzy sets

\[ A : X \rightarrow [0, 1] \]

Discrete fuzzy sets

\[ A : \{x_1, x_2, x_3, \ldots, x_s\} \rightarrow [0, 1] \]

Digital fuzzy sets

If a discrete-universal membership function can take only a finite number \( n \geq 2 \) of distinct values, then we call this fuzzy set a digital fuzzy set.

\[ A : \{x_1, x_2, x_3, \ldots, x_s\} \rightarrow \{0, \frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, \ldots, \frac{n-2}{n-1}, 1\} \]
Fuzzy Sets and Fuzzy Techniques
Joakim Lindblad
Outline
Introduction
About the course
Chapter one
(Crisp) Set Theory
1.2 Crisp sets: An overview
Fuzzy sets
Fuzzy sets of different types
Basic concepts and terminology
1.3 Fuzzy sets: Basic types
1.4 Fuzzy sets: Basic concepts

Fuzzy sets

**Universal set**

$X$ is the universe of discourse, or universal set, which contains all the possible elements of concern in each particular context of applications.

**Membership function** (compare Characteristic function)

The membership function $M$ maps each element of $X$ to a membership grade (or membership value) between 0 and 1.

A fuzzy set $M$, in the universal set can be presented by:

- list form,
- rule form,
- membership function form.

Fuzzy setsof different types

Fuzzy setsof different levels

Basic concepts and terminology

1.3 Fuzzy sets: Basic types
1.4 Fuzzy sets: Basic concepts

Fuzzy sets of different types

Interval-valued fuzzy sets

$A : X \rightarrow \mathcal{E}([0, 1])$

Fuzzy sets of type 2

$A : X \rightarrow \mathcal{F}([0, 1])$

L-fuzzy sets

$L$ is any partially ordered set

$A : X \rightarrow L$
Fuzzy sets of different levels

Also the domain of the membership function may be fuzzy. Fuzzy sets defined so that the elements of the universal set are themselves fuzzy sets are called level 2 fuzzy sets.

\[ A : \mathcal{F}(X) \rightarrow [0, 1] \]

Using a universal set containing level 2 fuzzy sets, we similarly get level 3 fuzzy sets, etc.

We will, however, stick to ordinary fuzzy sets, of type 1 and level 1.

Basic concepts and terminology

A crossover point of a fuzzy set is a point in \( X \) whose membership value to \( A \) is equal to 0.5.

The height, \( h(A) \) of a fuzzy set \( A \) is the largest membership value attained by any point. If the height of a fuzzy set is equal to one, it is called a normal fuzzy set, otherwise it is subnormal.

Basic concepts and terminology

The support of a fuzzy set \( A \) in the universal set \( X \) is a crisp set that contains all the elements of \( X \) that have nonzero membership values in \( A \), that is,

\[ \text{supp}(A) = \{ x \in X \mid A(x) > 0 \} \]

A fuzzy singleton is a fuzzy set whose support is a single point in \( X \).

An \( \alpha \)-cut of a fuzzy set \( A \) is a crisp set \( ^\alpha A \) that contains all the elements in \( X \) that have membership value in \( A \) greater than or equal to \( \alpha \).

\[ ^\alpha A = \{ x \mid A(x) \geq \alpha \} \]

A strong \( \alpha \)-cut of a fuzzy set \( A \) is a crisp set \( ^{\alpha +} A \) that contains all the elements in \( X \) that have membership value in \( A \) strictly greater than \( \alpha \).

\[ ^{\alpha +} A = \{ x \mid A(x) > \alpha \} \]
Basic concepts and terminology

We observe that the strong $\alpha$-cut $0^{+}A$ is equivalent to the support $\text{supp}(A)$.

The 1-cut $1A$ is often called the core of $A$.

**Note!** Sometimes the highest non-empty $\alpha$-cut $\beta(A)A$ is called the core of $A$. (in the case of subnormal fuzzy sets, this is different).

The word kernel is also used for both of the above definitions. (Total confusion!)

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Basic concepts and terminology

A fuzzy set $A$ defined on $\mathbb{R}^n$ is convex iff

$$A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(A(x_1), A(x_2)),$$

for all $\lambda \in [0, 1], x_1, x_2 \in \mathbb{R}^n$ and all $\alpha \in [0, 1]$.

Or, equivalently, $A$ is convex if and only if all its $\alpha$-cuts $\alpha A$, for any $\alpha$ in the interval $\alpha \in (0, 1)$, are convex sets.

Any property that is generalized from classical set theory into the domain of fuzzy set theory by requiring that it holds in all $\alpha$-cuts in the classical sense is called a cutworthy property.

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1.3 Fuzzy sets: Basic types

The ordering of the values of $\alpha$ in $[0, 1]$ is inversely preserved by set inclusion of the corresponding $\alpha$-cuts as well as strong $\alpha$-cuts. That is, for any fuzzy set $A$ and $\alpha_1 < \alpha_2$ it holds that $\alpha_2 A \subseteq \alpha_1 A$.

All $\alpha$-cuts and all strong $\alpha$-cuts for two distinct families of nested crisp sets.

The set of all levels $\alpha \in [0, 1]$ that represent distinct $\alpha$-cuts of a given fuzzy set $A$ is called a **level set** of $A$.

$$\Lambda(A) = \{\alpha \mid A(x) = \alpha \text{ for some } x \in X\}.$$
1.4 Fuzzy sets: Basic concepts

- alpha-cut and strong alpha-cut
- The level set $\Lambda$ (subset range $([0,1])$)
- Subset ordering of alpha cuts (inverse to alpha) $\subsetneq$ nested crisp sets
- New way to define FS, set of alpha cuts
- Support, core, height, normal & subnormal
- Convexity (on $\mathbb{R}^n$) (not convex function!) [proof]
- Cutworthy and strong cutworthy properties (holds for all alpha cuts)

- Standard fuzzy set operations
  - Complement, equilibrium points
  - Union & intersection
  - Lattice: De Morgan lattice/algebra - No law of contr. and excl. middle
  - Set inclusion (and equality)
  - Scalar cardinality (sigma count!)
  - Degree of subsethood
- Notation with slash: $A = A(x_1)/x_1 + A(x_2)/x_2 = \sum \ldots$ or $\int \ldots$ for continuous sets
- Geometric interpretation (prob. distr. sum to 1)