1 Motivation

a) What is the underlying idea of the fuzzy set theory?

Answer:
Graded belongingness to sets. A point is not only in the set or not, but belongs to it to some extent. (1p)

b) Give a few important reasons to apply the concepts of the fuzzy set theory? When are approaches based on fuzzy set theory the most useful (or needed)?

Answer:
Because information is never certain. Concepts of the real world are vague. E.g., in image processing, in the presence of noise, blurring, limited resolution etc. (2p)

c) Probability theory has existed since, at least, 1930. Why do we need fuzzy set theory in addition? Is it not all covered by probability theory? What are the main differences between these two theories? Can we just replace probability theory with the theory of fuzzy sets? or do the two theories complement each other? Give explanation/examples for your opinion.

Answer:
Probability deals with events that either happen or don’t happen. Fuzziness deals with non-sharp boundaries, which may exist in a totally static situation without any randomness at all. The two theories complement each other in the sense that we
can talk about the probability that a fuzzy event occurs, and also, events that occur to some extent.

2 Basic notions

A fuzzy set $S$ on $\mathbb{Z}$ is shown to the right.

a) What is the Support of $S$?
b) Height of $S$?
c) Is $S$ convex?
d) Is $S$ normal?
e) What is the cardinality of $S$?
f) What is the level set of $S$?
g) What is the $\beta$-cut at level 0.5?
h) What is the non-specificity of $S$ (in bits)?

Answer:
{2,...,9}, 0.8, yes, no, 3.4, \{0.2,0.6,0.8\}, \{6,...,9\}, $\frac{1}{\mu(A)} \int h(A) \log_2|\alpha A| = 1/0.8 \cdot (0.2 \cdot 3 + 0.4 \cdot 2 + 0.2 \cdot 0)$

3 Extension principle

Let $A$ and $B$ be fuzzy sets defined on $\mathbb{Z}$, with membership functions given by:

$A(x) = 0.5/(-1) + 0.8/0 + 1/1 + 0.2/2 \quad$ and \\
$B(x) = 0.5/1 + 1/2 + 0.5/3 + 0.8/4$

Let a function $f : X \times X \rightarrow X$ be defined for all $x_1, x_2 \in X$ by $f(x_1, x_2) = x_1 \cdot x_2$. Calculate $f(A, B)$.

Answer:
$[f(A, B)](z) = \sup_{x_1,x_2} \min(A(x_1), B(x_2))$
$0.5/(-4)+0.5/(-3)+0.5/(-2)+0.5/(-1)+0.8/0+0.5/1+1/2+0.5/3+0.8/4+0.2/6+0.2/8$

4 Fuzzy $c$-means

List some properties of the fuzzy $c$-means algorithm. What kind of algorithm is it? (I.e., what is it good for?) What parameters are required to set? What is the output? What is required to use it for multi-dimensional data? Give a brief description on how the algorithm works.

Answer:
It is a clustering algorithm. It groups data into clusters in an unsupervised manner, without any need for training data or seed-points. Parameters: Exponent controlling level of fuzziness in the output and $c$ – number of clusters. Output is a fuzzy partitioning of the
data, where the total memberships of one point to all the clusters sum up to one. Clusters may overlap. The method works on multi-dimensional data without any problems, so nothing in addition is required (trick question). The algorithm starts with a (random) partitioning of the space, cluster centers are computed (center of gravity) and a new partitioning is computed where the membership is defined based on the distance to a cluster center. This process of computing cluster centers, and partitioning the space, is repeated until stability.

5 Properties of fuzzy sets

a) How can crisp concepts, defined for crisp sets, be generalize to fuzzy sets? Write at least two methods.

Answer:
By extension principle, by integration over alpha-cuts, by fuzzification of concepts of the definition...

b) How are area and perimeter (usually) defined for fuzzy spatial sets?

Answer:
By integration over alpha cuts.

c) Try to explain why the isoperimetric inequality does not hold for fuzzy sets using the standard definitions of area and perimeter.

Answer:
Isoperimetric ineq. says: \( \frac{4\pi \cdot \text{Area}}{\text{Perim}^2} \leq 1 \). This holds for crisp objects. However not for fuzzy objects, where a fuzzy disk (unintuitively) is more compact than a crisp disk(!). The reason for this is the non-linear combination in the inequality. Both Perim, and Area scale linearly with the height of a fuzzy set, so by reducing the height, the fraction \( \frac{\text{Area}}{\text{Perim}^2} \) grows (without limit).

d) Örjan Smedby is facing the task of making measurements of very thin bone structures in images of limited resolution. Do you think that fuzzy techniques could help him? Give arguments why and, if so, in what ways?

Answer:
Yes! Fuzzy techniques are useful when spatial resolution is limited. Feature estimates (measurements) are more precise if estimated from a fuzzy, rather than a crisp image. It is, in addition, often easier to perform a fuzzy segmentation in a robust way than a crisp one.

6 Operations on fuzzy sets

a) Definitions for set complement, intersection and union are based on an axiomatic approach. Write down the axiomatic skeleton (main axioms) for defining fuzzy intersections.

Answer:
\( i(a, 1) = a, b \leq d \Rightarrow i(a, b) \leq i(a, d), i(a, b) = i(b, a), i(a, i(b, d)) = i(i(a, b), d) \)
b) The so-called standard set operations have a special position among set operations on fuzzy set. Give definitions of standard complement, intersection, and union.

Answer:
\[
[c(A)](x) = 1 - A(x), \ [i(A, B)](x) = \min(A(x), B(x)), \ [u(A, B)](x) = \max(A(x), B(x))
\]

(1p)

c) Give at least two properties that make the standard intersection and union “special”?

Answer:
Cutworthiness, largest possible intersection and smallest possible union, prevents compound of errors, only idempotent ones.

(2p)

d) What is a dual triple? Give three examples of dual triples.

Answer:
Satisfies DeMorgan’s laws. See book for three examples.

(2p)

7 Approximate reasoning

A popular method for approximate reasoning is the method of interpolation. Use the method of interpolation to infer a conclusion from the following situation. Draw the
fuzzy conclusion set $B'$ and explain the steps that you took to find that result.

| Rule 1: If $X$ is $A_1$, then $Y$ is $B_1$ |
| Rule 2: If $X$ is $A_2$, then $Y$ is $B_2$ |
| Fact: $X$ is $A'$ |
| Conclusion: $Y$ is $B'$ |

Remember to hand in this page together with your solution, if you did draw in the figure to the right.

Answer:
See Lecture notes, end of Lecture 10. $B' = B'_1 \cup B'_2$, where $B'_i$ is $B_i$ truncated at the height $h(A' \cap A_i)$. (3p)

8 Fuzzy control

List the different parts of a fuzzy controller. Describe the roles and purposes of the different parts.

Answer:
Fuzzifier, Rule base, Inference engine, Defuzzifier. Fuzzifier fuzzifies the crisp input to better reflect its real uncertainty. The rule base consist of fuzzy if-then rules covering the (granulated) input space. The inference engine infers an output from the input using the rules of the rule base (as in Q 7). The fuzzy output of the inference is defuzzified (to a point) to give a crisp control value out. (4p)
9 Fuzzy connectedness

a) Describe the following fuzzy connectedness concepts:
   (a) Fuzzy adjacency
   (b) Fuzzy affinity
   (c) Fuzzy connectedness
   (d) A path, and the strength of a path

   Answer:
   Local spatial hanging-togetherness
   Local spatio-membership hanging-togetherness
   Global spatio-membership hanging-togetherness
   A sequence of adjacent spels, strength is minimum affinity along the path.

b) How can global hanging-togetherness be computed from local hanging-togetherness using the method of fuzzy connectedness?

   Answer:
   By setting the global hanging-togetherness between two spels to be the strength of the strongest path connecting them, where the strength of a path is based on local hanging-togetherness (see above). This is efficiently computed using dynamic programming.

These are very condensed answers and in general too brief to give full points!