Uniform Polyhedra

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Humans have always been fascinated by regular patterns and bodies. The shapes can be enjoyed without understanding, but below there is some definitions and history on uniform polyhedra. The first uniform polyhedra (for example the cube) was discovered very early in the depths of time, but the work of finding all was, perhaps surprisingly, not finished until 1975.

THE POLYHEDRA SURFACE PATCHES

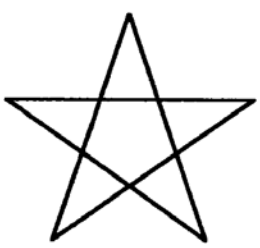
The polyhedra has symmetrical facets, therefore we start in two dimensions.

In a **regular polygon**, all sides are equally long and all angles are also equal. In a **convex regular polygon** no sides cross each other (and all inner angles are ≥ 60 degrees). These figures are named after the Greek numbers plus -gon and those that can occur in the uniform polyhedra are:

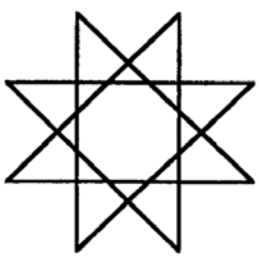
trigon (or equilateral triangle, three sides, symbol {3}
tetragon (or square), four sides {4}
pentagon five sides {5}
hexagon six sides {6}
heptagon seven sides {7}
octagon eight sides {8}
decagon ten sides {10}
dodecagon twelve sides {12}

In a **non-convex regular polygon** all sides are still of equal length and all angles are equal, but the sides are allowed to cross each other, so the figures are star-like. The smallest number of sides is five. There are often several different stars with the same number of sides, depending on how many corners a side "passes" before reaching a star point. However, there are also cases for which there is not a single star, for example there is no regular star-polygon with six sides. The regular stars have, again, names after the Greek numbers plus -gram and all those existing with up to ten sides are listed below. The symbol $\{n/(p+1)\}$ shows the number of sides n and the number of corner p passed.

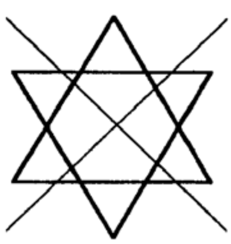
pentagram five sides, one corner passed $\{\frac{5}{2}\}$
heptagram seven sides, one corner passed $\{\frac{7}{2}\}$
heptagram seven sides, two corners passed $\{\frac{7}{3}\}$
octagram eight sides, two corners passed $\{\frac{8}{3}\}$
decagram ten sides, two corners passed $\{\frac{10}{3}\}$



$\{\frac{5}{2}\}$



$\{\frac{8}{3}\}$



The "Star of David", right, is not a "hexagram", it consists of two overlapping triangles.

DEFINTIONS

A **uniform polyhedron** is a shape fulfilling the two conditions:
• All its surface facets are regular polygons
• All vertices are equal

A **regular polyhedron** also fulfils:
• All surfaces factes are equal

REGULAR POLYHEDRA

The Platonic solids are defined as the convex regular polyhedra. *[Plato, Greece, ≈428 - ≈348 BCE]* There are five of them. They are named after the Greek numbers for the numbers of facets. The Schläfli symbol {...} describes which polygons meet in each vertex.

Tetrahedron, four trigon facets, three meet in each vertex {3 3 3}
Known in old Egypt (long before Plato)
Octahedron, eight trigon facets, four meet in each vertex {3 3 3 3}
Known in old Egypt
Icosahedron, twenty trigon facets, five meet in each vertex {3 3 3 3 3}
Known in old Egypt
Hexahedron, six tetragon facets, three meet in each vertex {4 4 4}
Known since the beginning of humanity
Dodecahedron, twelve pentagon facets, three meet in each vertex {5 5 5}
Known Etruria (present-day Italy) before 500 BCE

There are four non-convex regular polyhedra. Polyhedra can be non-convex in two ways: the facets can be star polyhedra or the vertex where the surfaces meet can be star-shaped. Or both.

Kepler's star polyhedra where discovered by Kepler *[Johannes Kepler, Germany, 1571-1630]* and published in 1619. Both have pentagram facets and are thus non-convex in the first way.

Great star dodecahedron, twelve pentagram facets, three meet in each vertex $\{\frac{5}{2}\frac{5}{2}\frac{5}{2}\}$

Small star dodecahedron, twelve pentagram facets, five meet in each vertex $\{\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}\frac{5}{2}\}$

Poinsot's star polyhedra where discovered by Poinsot *[Louis Poinsot, France, 1777–1859]* and published in 1810. The facets are trigons in one case and pentagons in the other. The vertices are both star-shaped in the $\left\{\frac{5}{2}\right\}$ configuration.

Great icosahedron, twenty trigon facets, five meet in each vertex {3 3 3 3 3}*
Great dodecahedron, twelve pentagon facets, five meet in each vertex {5 5 5 5 5}*

UNIFORM POLYHEDRA

The **Archimedean solids** are defines as the convex uniform polyhedra (that are not regular) *[Archimedes, Greek Sicily, ≈285- 212 BCE]*. All facets are convex regular polygons, at least two different ones, and all vertices are equal. There are thirteen Archimedean solids. Archimedes did probably knew them, but they were forgotten and re-discovered by and by. A complete list was first published by Kepler, in "Harmionices Mvndi" 1619 where the star polyhedra are also found. His names for the Archimedean polyhedra are still mostly used, even if new names are sometimes invented by re- re-discoverers. A blatant example is the "Bucky-ball" that is simply a truncated icosahedron. The symbol {...} shows what polygons meet in which order.

Cubeoctahedron (the Greeks called it *dymaxion*) {3 4 3 4}
Icosidodecahedron {3 5 3 5}

Truncated tetrahedron {3 6 6}
Truncated octahedron {4 6 6}
Truncated icosahedron {5 6 6}
Truncated hexahedron {3 8 8}
Truncated dodecahedron {3 10 10}

Truncated cubeoctahedron {4 6 8}
Truncated icosidodecahedron {4 6 10}

Rhombicubeoctahedron {3 4 4 4}
Rhombicosidodecahedron {3 4 5 4}

Snub cube {3 3 3 3 4}
Snub dodecahedron {3 3 3 3 5}

Prisms and **anti-prisms** are also convex uniform polyhedra, which Kepler was the first to realise. A prism is limited by two convex polygons and tetragons on the facets, {4 4 n}, where $n \geq 3$. An anti-prism is limited by two convex polygons and trigons, {3 3 3 m}, where $m \geq 3$. Examples on show are:

Heptagonal prism {4 4 7}
Heptagonal anti-prism {3 3 3 7}

There are in addition, 53 **non-convex uniform** polyhedra. They started to be discovered in the 1800s. Badoureau discovered 37 *[Albert Badoureau, France, 1853-1923]* and Pitsch 18 *[Johann Pitch, Austria]*. They did not know about each other, but both published in 1881. Together they had found 41 uniform polyhedra. Then nothing happened for a while. In the years 1930-32 Coxeter *[“Donald” Coxeter, Canada, 1907-2003]* and Miller *[Jeffrey Charles Percy Miller, Britain, 1906–1981]* discovered the remaining twelve, but as they had no proof they had found all they did not publish. Longuet-Higgins *[Michael S. Longuet-Higgins, Britain, 1925--]* independently discovered eleven of the twelve 1940-42. Both Miller and Longuet-Higgins worked at Cambridge University in England, but they had no idea they were both interested in uniform polyhedra until they discovered this by chance in 1952. This led to a joint publication of the twelve by Coxeter, Miller, and Longuet-Higgins in 1954. Finally, in 1975 Skilling *[John Skilling, Britain]*, also from Cambridge could prove that the set on uniform polyhedra was complete. The work that started with a cube at the dawn of humanity was finished! Most of the uniform polyhedra are present in the exhibition, three more in room 2142, but a few are too difficult. In each vertex of “Miller’s Monster” two trigons, four tetragons and two pentagrams meets, and the vertex itself is non-convex. The complete set *is* exhibited at the Technical museum in London.

Non-convex prisms, **anti-prims** and **star-prisms**, all based on star polygons, are also uniform polyhedra, just like convex prisms. Their configurations are{4 4 $\frac{n}{m}$}, {3 3 3 $\frac{n}{m}$}, and {3 3 3 $\frac{n}{m}$}*, respectively. These are knows since the 1800s. The prism is straight, the anti-prism is turned a half-step and the star prism is turned one step. The examples in the exhibition have heptagrams as top and bottom:

Heptagrammal prism {4 4 $\frac{7}{3}$}
Heptagrammal anti-prism {3 3 3 $\frac{7}{3}$}

Heptagrammal star-prism {3 3 3 $\frac{7}{3}$}*

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