Anders Hast
University of Gävle
Creative Media Lab
Gävle, Sweden
aht@hig.se

GEOMETRY OF FIBERS WITH VARYING TOPOLOGY

INTRODUCTION. Anisotropic shading techniques can be used for materials like hair [4, 5] and silk [3]. Such materials have a surface consisting of very small fibers having a main direction, i.e. the tangent direction of the surface. Each fiber can be viewed as a cylinder. Poulin and Fournier [3] proposed an illumination model for such materials using a flat topology only. If the fibers have a varying topology [1] that allows fibers to climb on other fibers as will be the case when for example a silk thread is winded around the body of a torus then the hiding and shadowing must be computed for three different possible cases. This model assumes that the cylinders to the left and right are at equal height compared to the cylinder in the middle. Figure 1 shows how threads are climbing on each other.

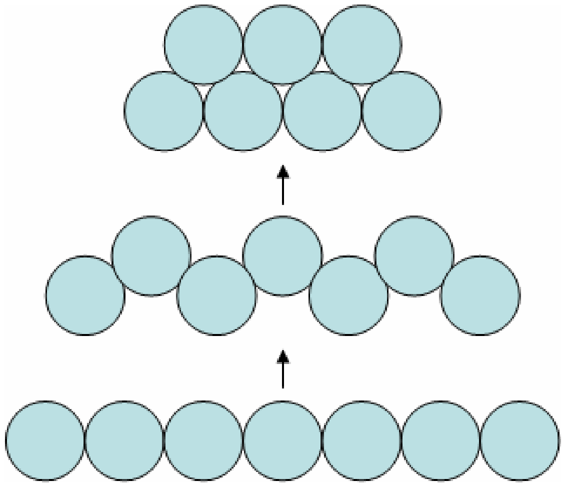

Fig. 1 The fibers in the flat topology (in the bottom) starts to climb on its neighbors (in the middle), creating a new topology and finally a flat topology is obtained (on the top) when the fibers have climbed as high as possible.

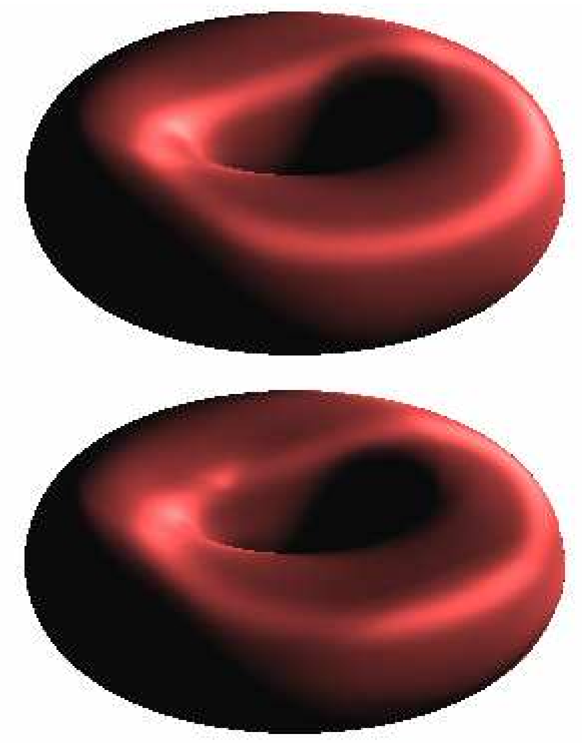

Figure 2 Flat topology (in the top) and varying topology (in the bottom).
Visible differences are apparent.

Figure 2 shows the difference between using the varying topology and just assuming a flat topology everywhere. The varying topology is a more correct model and visible differences are clearly apparent. In the rest of the paper the necessary equations for computing the geometry of the fibers are presented. How to compute the actual shading is presented in [1].

INTERSECTIONS. The computation of intersections in the hiding and shadowing of fibers with varying topology will always involve the rotation of a vector with unit length. These computations can be performed efficiently on modern hardware with vector operation support, since rotation is just a matrix product. There are three different cases that must be determined for shading.

Figure 3 shows the projected view vector and the hiding and shadowing. It is necessary to compute the two intersection points $\mathbf{I_1}$ and $\mathbf{I_2}$. This can be done using the equations for ray-sphere intersection [1].

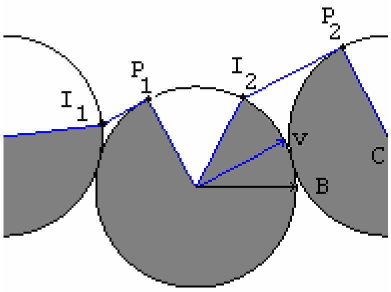

Fig. 3 The geometry of fibers showing intersection points.

The view vector **v** and the vector in the direction to the light source **l** are both projected onto the plane spanned by the normal and the bi-tangent. In this space it is possible to compute the intersection points. It will be shown how the intersections can be computed using an arbitrary vector **v**, which can be defined as

$$\mathbf{v} = [\cos\theta, \sin\theta] \tag{1}$$

where θ is the angle of the vector with respect to the bi-tangent **B** on the unit circle that is represented by each fiber. The points $\mathbf{P_1}$ and $\mathbf{P_2}$ in figure 3 is therefore

$$\mathbf{P_1} = [-\sin\theta, \cos\theta], \quad \mathbf{P_2} = \mathbf{C} + [-\sin\theta, \cos\theta] \tag{2}$$

where **C** is

$$\mathbf{C} = 2[\cos\phi, \sin\phi] \tag{3}$$

and ϕ is the angle between the point **C** and the bi-tangent **B**. The intersection points $\mathbf{I_1}$ and $\mathbf{I_2}$ can be computed by using the equation for ray sphere-intersection [1]. If the cylinder disc is centered around the origin with the radius 1, in some arbitrary scale and the direction of the ray is represented by **v** and the start point is **P,** then the ray is

$$\mathbf{R}(t) = \mathbf{P} + t\mathbf{v} \tag{4}$$

Hence the intersection point can be computed using these intermediate computations

$$\begin{aligned} a &= v^2 \\ b &= 2(\mathbf{P} \bullet \mathbf{v}) \\ c &= P^2 - 1 \end{aligned} \tag{5}$$

The intersection will be at

$$t = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{6}$$

The parameter *t* is finally put into equation 4, in order to obtain the intersection points. The final equation is rather complex when expressed in terms of θ and ϕ.

RESULTS. The final formulas will be presented in this section. Use equation 5 and 6 to compute $\mathbf{I_1}$ in figure 3, then

$$\mathbf{I_1} = \begin{bmatrix} -\sin\theta - 2\cos^2\theta\cos\varphi + 2\sin\theta\cos\theta\sin\varphi - \cos\theta\sqrt{q} \\ \cos\theta - 2\sin\theta\cos\theta\cos\varphi + 2\sin^2\theta\sin\varphi - \sin\theta\sqrt{q} \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} q = 1 + 4\cos^2\theta\cos^2\varphi - 8\cos\theta\sin\theta\cos\varphi\sin\varphi + 4\sin^2\theta\sin^2\varphi \\ - 4\cos^2\varphi - 4\sin^2\varphi - \sin^2\theta - \cos^2\theta + 4\sin\theta\cos\varphi + 4\cos\theta\sin\varphi \end{aligned} \tag{8}$$

Similar equations will be obtained for the other two cases. Obviously there are some simplifications that can be done since it is possible to rearrange the terms using the pythagorean identity. Moreover, we can arrange the terms in such way that a geometrically interesting relation is clearly visible. The equation for computing point $\mathbf{I_1}$ then becomes

$$\mathbf{I}_1 = 2[-\cos\phi, \sin\phi] + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \tag{9}$$

where

$$\begin{aligned} \cos\varphi &= \sqrt{1-(1-2\sin\theta\cos\phi-2\cos\theta\sin\phi)^2} \\ \sin\varphi &= 1-2\sin\theta\cos\phi-2\cos\theta\sin\phi \end{aligned} \tag{10}$$

It can easily be shown that $[\cos\varphi, \sin\varphi]^T$ is a unit length vector. Furthermore it is obvious that this vector is rotated clockwise by the preceding matrix with an angle θ. The first term in equation 9 is just a translation. Similar equations can be derived for $\mathbf{I_2}$ as well as for the other cases that will occur, e.g. the whole middle cylinder can be hidden. Each equation will be a rotation with an angle θ of a vector with unit length. Moreover, for this case we have

$$\sin\varphi = 1-2\sin(\theta+\phi) \tag{11}$$

In the following equations only this latter formula will be shown since it is compact and the cosine can be computed from it.

The intersection point $\mathbf{I_2}$ in figure 3 lies on the middle cylinder and no translation is necessary. The final formula becomes

$$\mathbf{I}_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \tag{12}$$

where

$$\sin\varphi = 1+2\sin(\phi-\theta) \tag{13}$$

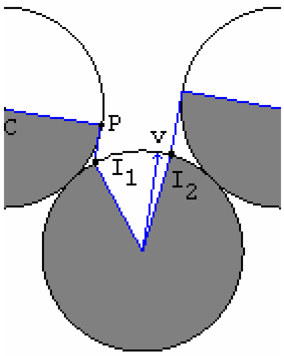

Fig. 4 The geometry of fibers showing intersection points for the second possible case.

When $\cos\theta < \sin\phi$ then we have the situation shown in figure 4 so that the left cylinder shadows the one in the middle. We can compute $\mathbf{I_1}$ as

$$\mathbf{I_1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \quad (14)$$

where

$$\sin\varphi = 2\sin(\theta + \phi) - 1 \quad (15)$$

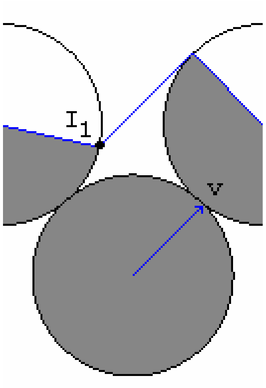

Fig. 5 The geometry of fibers showing intersection points
for the third case.

If the sum of the terms inside the square root involved in computing cos φ are negative then we have the case shown in figure 5. The intersection point can now be computed using

$$\mathbf{I_1} = 2[-\cos\phi, \sin\phi] + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \quad (16)$$

where

$$\sin\varphi = 1 - 2\sin(\theta)\cos(\phi) \quad (17)$$

Also note that we must be able to handle the case when the vector **v** is pointing in the direction to the left. So far the pictures in figure 3-5 have shown the vector pointing to the right. For these cases we get very similar computations.

CONCLUSIONS. The computation of intersections in the hiding and shadowing of fibers with varying topology involves a rotation of a vector with unit length. This vector depends on the other two angles involved, i.e. the view (or light source vector) and the vector in the direction to the center of the elevated cylinder. These equations can easily be implemented in modern graphics hardware to compute the geometry used for anisotropic shading of fibers with varying topology.

REFERENCES.

[1] A. Hast, A Model for Anisotropic Shading of Fibers with varying Topology
SCCG'06 full paper, 2006, pp 44-50.

[2] Lengyel E, Mathematics for 3D Game Programming & Computer Graphics, second ed. Charles River Media, 2004. pp. 146-148.

[3] Poulin, P, Fournier A, A model for anisotropic reflection. ACM SIGGRAPH Computer Graphics, Proceedings of the 17th annual conference on Computer graphics and interactive techniques, Volume 24 Issue 4, 1990 pp. 273–282.

[4] Stalling D, Zöckler M, Hege H-C, Fast Display of Illuminated Field Lines, IEEE Transactions on Visualization and Computer Graphics, 1997, Volume 3 , Issue 2 , pp. 118 - 128.

[5] Zöckler M, Stalling D and Hege H-C. Interactive visualization of 3D-vector fields using illuminated stream lines. In IEEE Visualization '96, pp 107-113, 1996.