A Rotation-Invariant Morphology for Shape Analysis of Anisotropic Objects and Structures

Cris L. Luengo Hendriks and Lucas J. van Vliet

Pattern Recognition Group, Delft University of Technology, the Netherlands.
cris@ph.tn.tudelft.nl

Abstract. In this paper we propose a series of novel morphological operators that are anisotropic, and adapt themselves to the local orientation in the image. This new morphology is therefore rotation invariant; i.e. rotation of the image before or after the operation yields the same result. We present relevant properties required by morphology, as well as other properties shared with common morphological operators. Two of these new operators are increasing, idempotent and absorbing, which are required properties for a morphological operator to be used as a sieve. A sieve is a sequence of filters of increasing size parameter, that can be used to construct size distributions. As an example of the usefulness of these new operators, we show how a sieve can be build to estimate a particle or pore length distribution, as well as the elongation of those features.

1 Introduction

When analyzing images without a preferred orientation, or images with an unknown orientation (as is the case, for example, of an image acquired after placing a sample randomly under a microscope), it is desirable to use rotation invariant operations. A rotation invariant operation yields an output that is independent of the orientation of the sample with respect to the sampling grid. There are three different ways of constructing rotation invariant operators:

- using a single isotropic operator (the kernel itself is rotation invariant),
- using a data-driven anisotropic operator (the kernel is anisotropic, but is oriented to the local gradient in the image), or
- by combining a set of anisotropic operators.

Non-rotationally invariant filters will almost certainly produce incorrect results if they are not aligned with the image under study, and an isotropic filter is often limited in its capabilities. Therefore, it is worthwhile to study rotation invariant operators based on anisotropic kernels.

For example, consider an isotropic morphological closing, which has a disk as the structuring element (we regard 2D images for now). If we apply such a filter to an image with dark objects, such as the microscopical image in Fig. 1, all dark objects smaller than the structuring element will be removed from the image. If we see the image as a landscape where the dark features are the valleys and the light ones the hills, as in Fig. 1, we can imagine the closing as filling up the valleys such that no valleys remain in which the structuring element cannot fit (see Fig. 2).

C. Arcelli et al. (Eds.): IWVF4, LNCS 2059, pp. 378–387, 2001.

This closing operation can be used as a sieve to detect features larger than a certain size. The problem is that this size is only determined by the smallest diameter of the features. To measure length, an anisotropic structuring element is required.

In this paper we will introduce new morphological operators based on isotropic structuring elements with a *lower* dimensionality than the image under study (and thus *anisotropic* in the space of the image). By dropping one or more dimensions, the structuring element gets some degrees of rotational freedom that allows it to align itself with the features in the image. By selecting the orientation that causes minimum or maximum response (pixel by pixel), we create a rotation-invariant operator. In the two-dimensional case, the structuring element would be one-dimensional, with one degree of rotational freedom. A closing in this new morphological framework would remove an object only if the line element could not fit. This would mean that its largest diameter (supposing convex objects) is smaller than the structuring element (see Fig. 3).

We will call this new morphological framework Rotation-Invariant Anisotropic (RIA) morphology. We can call it morphology because it satisfies the four principles of morphology [1]:

- Translation invariance,
- Compatibility under change of scale,
- Local knowledge, and
- Semi-continuity.

The first three principles are expressed as properties of the operators in Sect. 3, and proven elsewhere [2]. The principle of semi-continuity requires that the theory in the continuous world has an approximate counterpart in the discrete world, and is responsible for this theory to be applicable in practice [3]. Although the discretization of the operators presented here is beyond the scope of this article, it certainly is possible to apply these operators to discrete images.

In Sect. 4 we will apply the new closing and opening introduced here to do segmentation-free measurements using morphological sieves. Sieves are used to build multi-parameter (length, width, depth) size distributions that characterize the shapes of objects, structures or textures in grey-value images. These measured distributions can be used for image recognition or characterization, and are applicable in a wide variety of situations.

2 Definitions

In this paper we use the notation as specified in Table 1. We will use Greek characters (especially φ and θ) for rotation angles, and Latin characters (especially x and y) for translation vectors and image coordinates. f and g denote continuos functions $\mathbb{R}^N \to \mathbb{R}$ (the image being processed). Vectors are not distinguished typographically because it is obvious from the context which variables are vectors and which ones are scalars.

2.1 Dilation

A flat, isotropic structuring element D of radius r can be decomposed into (an infinite amount of) rotated line segments L_φ of length $\ell = 2r$. The dilation then becomes, with$\varphi \in [0, \pi)$,

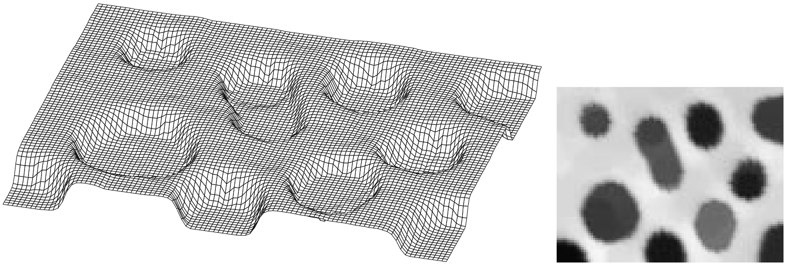

Fig. 1. A portion of the image ‘cermet’, after some processing.

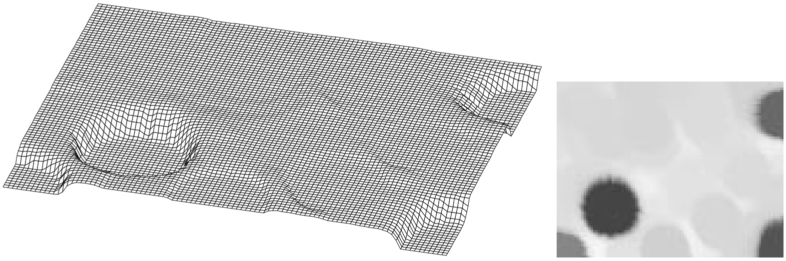

Fig. 2. The image from Fig. 1, after closing with a circular structuring element.

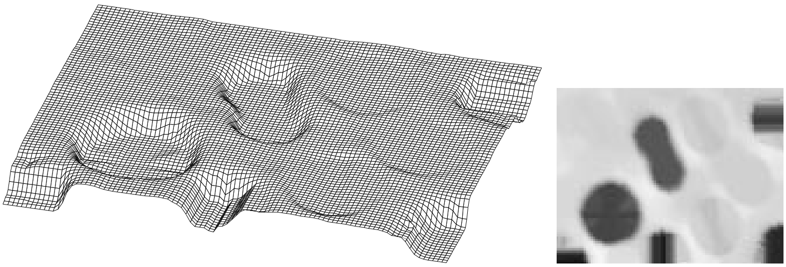

Fig. 3. Result of the new, rotation invariant, anisotropic morphological closing applied to the image in Fig. 1. Compare with the result of the isotropic closing in Fig. 2.

Table 1. Notation used in this paper

f_φ	rotation of f over an angle φ
f_x	translation of f over x: $f_x(t) = f(t-x)$
$f_{\varphi,x}$	rotation of f over an angle φ and then a translation over x
$S_b f$	scaling of f with a factor b: $S_b f(x) = f(\frac{x}{b})$
$\oplus$	Minkowski addition
$\check{S}$	transpose of S: $\check{S}(x) = S(-x)$
$\delta_S f$	dilation of f with S, any flat structuring element: $\delta_S f = f \oplus \check{S} = \bigvee_{x\in\check{S}} f_x$
$\varepsilon_S f$	erosion of f with structuring element S
$\gamma_S f$	opening of f with structuring element S
$\phi_S f$	closing of f with structuring element S
$A \triangleq B$	definition: "Let A be defined as B"
$A \doteq B$	equality by definition: "A is equal to B by definition"

$$\delta_D f \doteq f \oplus D = f \oplus \bigcup_\varphi L_\varphi = \bigvee_\varphi (f \oplus L_\varphi) \doteq \bigvee_\varphi \bigvee_{x\in L_\varphi} f_x \quad . \tag{1}$$

Note that we ignore the transpose operation since $\check{D} = D$ and $\check{L}_\varphi = L_\varphi$.

Based on this, we define a new morphological operator, which we will call RIA dilation, and denote with the symbol $\delta^\triangleleft$,

$$\delta_L^\triangleleft f \triangleq \bigwedge_\varphi \bigvee_{x\in L_\varphi} f_x \doteq \bigwedge_\varphi \delta_{L_\varphi} f \quad . \tag{2}$$

This operator takes the maximum of the image over a line segment rotated in such a way as to minimize this maximum. Figure 4 gives an example of the effect that the operator has on an object boundary. Note that a convex object boundary is not changed, but a concave one is.

We like to compare this dilation operator with a train running along a track. The train wagons (which are joined at both ends to the track) require some extra space at the inside of the curves. This dilation, applied to a train track, and using a structuring element with the length of the wagons, reproduces the area required by them.

2.2 Erosion

RIA erosion is defined as the dual of the RIA dilation, and will be denoted with the symbol $\varepsilon^\triangleleft$.

$$\varepsilon_L^\triangleleft f \triangleq -\delta_L^\triangleleft(-f) \doteq -\bigwedge_\varphi \bigvee_{x\in L_\varphi} (-f_x) = \bigvee_\varphi \bigwedge_{x\in L_\varphi} f_x \doteq \bigvee_\varphi \varepsilon_{L_\varphi} f \quad . \tag{3}$$

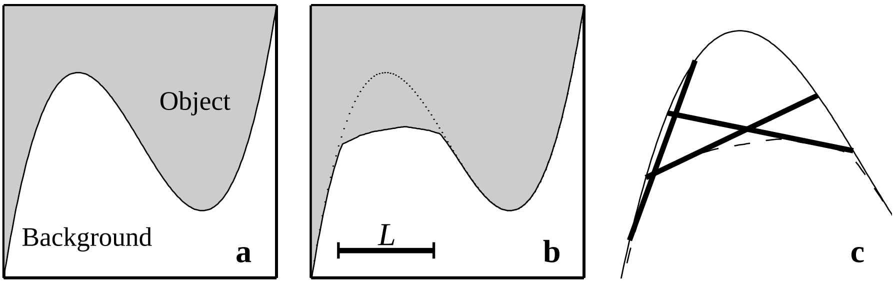

Fig. 4. Effect of the RIA dilation on an object boundary. **a:** The original boundary. **b:** The boundary after the dilation, together with the line segment used as a structuring element. **c:** Construction of the dilated object boundary.

2.3 Closing

The closing is usually defined as a dilation followed by an erosion,

$$\phi_D f \doteq \varepsilon_D \delta_D f \quad . \tag{4}$$

However, it is easier to understand (and modify) if we see it as the maximum of the image over the support of the structuring element D, after shifting it in such a way that it minimizes this maximum, but still hits the point t at which the operation is being evaluated, (see Fig. 5a). Or, in other words, the 'lowest' position we can give D by shifting it over the 'landscape' defined by the function f:

$$\phi_D f = \bigwedge_{x \in D} \bigvee_{y \in D_x} f_y \quad \left[= \bigwedge_{x \in D} \left(\bigvee_{y \in D} f_y \right)_x = \varepsilon_D \delta_D f \right] \quad . \tag{5}$$

In accordance to this, we define a new morphological operation, RIA closing, as the 'lowest' position we can give the linear structuring element L, by shifting and rotating it over the 'landscape' f, such that it still hits the point x being evaluated (see Fig. 5b). It will be denoted by $\phi^{\lhd}$, and defined by

$$\phi_L^{\lhd} f \triangleq \bigwedge_{\varphi} \bigwedge_{x \in L_\varphi} \bigvee_{y \in L_{\varphi,x}} f_y \quad , \tag{6}$$

which is analogous to the definition of the RIA dilation, where we also changed the disk for a line, and added a minimum over the orientation of that line. As it turns out, this is the same as the minimum of the closings, at all orientations, with a line segment as structuring element,

$$\phi_L^{\lhd} f \doteq \bigwedge_{\varphi} \bigwedge_{x \in L_\varphi} \left(\bigvee_{y \in L_\varphi} f_y \right)_x = \bigwedge_{\varphi} \varepsilon_{L_\varphi} \delta_{L_\varphi} f \doteq \bigwedge_{\varphi} \phi_{L_\varphi} f \quad , \tag{7}$$

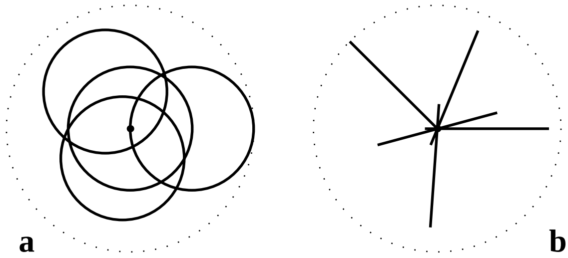

Fig. 5. a: The closing with an isotropic structuring element (disk) is determined by shifting the disk in such a way that it still hits the point being evaluated, and minimizes the supremum of the image over its support. **b:** The RIA closing is determined by shifting and rotating the line segment in such a way that it still hits the point being evaluated, and minimizes the supremum of the image over its support.

but not equal to a RIA dilation followed by a RIA erosion.

We will show elsewhere [2] that this transformation is increasing, idempotent and extensive, and therefore we can call it an algebraic closing [4]. Moreover, Matheron has shown that any intersection of morphological closings is an algebraic closing [5]. We can interpret $\bigwedge_{\varphi} \phi_{L_\varphi} f$ as the intersection of an infinite series of closings, in which case the increasingness, idempotence and extensivity are proven by Matheron. For previous work using rotated line segments see Soille [6].

2.4 Opening

The RIA opening is defined as the dual of the RIA closing, and denoted by the symbol $\gamma^{\triangleleft}$.

$$\gamma_L^{\triangleleft} f \triangleq -\phi_L^{\triangleleft}(-f) \doteq -\bigwedge_{\varphi} \bigwedge_{x \in L_\varphi} \bigvee_{y \in L_{\varphi,x}} (-f_y) = \bigvee_{\varphi} \bigvee_{x \in L_\varphi} \bigwedge_{y \in L_{\varphi,x}} f_y \doteq \bigvee_{\varphi} \gamma_{L_\varphi} f \quad . \tag{8}$$

2.5 Extension to Higher Dimensionalities

Until now we have only talked about operations on two-dimensional images. However, it is very easy to extend the RIA morphology to higher dimensionalities. For example, in the 3D case, it would be possible to have structuring elements with either one or two dimensions (i.e. a disk or a line segment); both have two degrees of rotational freedom. A closing with these two structuring elements can be used to measure the first and second largest diameters of the (convex) object: the line segment can not fit if it is longer than the largest diameter; the disk can not fit it is wider than the second largest diameter. To measure the smallest diameter, the isotropic closing would be used.

3 Properties

Properties that are valid for all operators are only specified for the RIA dilation. Properties mentioned only for the RIA closing are by duality also true for the RIA opening but not for the dilation or erosion.

Property 1. *Translation invariance:*

$$\delta_L^{\triangleleft} f_x = (\delta_L^{\triangleleft} f)_x$$

Property 2. *Compatibility under change of scale:*

$$S_b \delta_L^{\triangleleft} f = \delta_{b \cdot L}^{\triangleleft} S_b f$$

The result of the operation is scaled by b if both the image and the structuring element are scaled by b.

Property 3. *Local knowledge:*

$$W_1 \cdot \delta_L^{\triangleleft}(W_2 \cdot f) = W_1 \cdot \delta_L^{\triangleleft} f$$

This property simply states that the result of the operator inside some window W_1 is independent of the image outside some other window W_2. This implies that $W_1 \subset W_2$.

These first three properties are the cornerstones of morphology, without which it is not possible to define shape. Together with the principle of semi-continuity, they are the requirements for operators to belong to morphology.

Property 4. *Rotation invariance:*

$$\delta_L^{\triangleleft} f_\theta = (\delta_L^{\triangleleft} f)_\theta$$

Rotation invariance of the RIA morphology is a key property, necessary for the correct analysis of images with an unknown orientation, or images without a single dominant orientation.

Property 5. *Contrast invariance:*

$$\delta_L^{\triangleleft}(c \cdot f) = c \cdot \delta_L^{\triangleleft} f$$

This property can be taken further, by stating that both the RIA dilation and the RIA closing commute with any anamorphoses (which is defined as an increasing and continuous mapping $\mathbb{R} \to \mathbb{R}$) [1].

Property 6. *Increasingness:*

$$f \leq g \quad \Longrightarrow \quad \delta_L^{\triangleleft} f \leq \delta_L^{\triangleleft} g$$

Property 7. *Extensivity / anti-extensivity:*

$$\varepsilon_L^{\triangleleft} f \leq f \leq \delta_L^{\triangleleft} f$$
$$\gamma_L^{\triangleleft} f \leq f \leq \phi_L^{\triangleleft} f$$

Property 8. *Extended extensivity:*

$$\varepsilon_L^{\triangleleft} f \leq \gamma_L^{\triangleleft} f \leq f \leq \phi_L^{\triangleleft} f \leq \delta_L^{\triangleleft} f$$

Property 9. *Idempotence:*

$$\phi_L^{\triangleleft} \phi_L^{\triangleleft} f = \phi_L^{\triangleleft} f$$

Property 10. *Absorption:*

$$\ell_1 \geq \ell_2 \quad \Longrightarrow \quad \begin{cases} \phi_{L^{(2)}}^{\triangleleft} \phi_{L^{(1)}}^{\triangleleft} f = \phi_{L^{(1)}}^{\triangleleft} f \\ \phi_{L^{(1)}}^{\triangleleft} \phi_{L^{(2)}}^{\triangleleft} f = \phi_{L^{(1)}}^{\triangleleft} f \end{cases}$$

Where $L^{(i)}$ is a linear structuring element with length ℓ_i.

This property states that applying a RIA closing at a large scale to the result of the RIA closing at a smaller scale yields the same results as applying it to the original image. Furthermore, applying other RIA closings at smaller scales after that has no effect.

Note that idempotence is a special case of absorption, where $\ell_1 = \ell_2$. Also, the comutativity of the RIA closing follows from the absorption property, since only the largest-scale operator influences the result, independently from the order in which they are applied.

Property 11. *Sieving:*

$$\ell_1 \leq \ell_2 \quad \Longrightarrow \quad \phi_{L^{(1)}}^{\triangleleft} f \leq \phi_{L^{(2)}}^{\triangleleft} f$$

The sieving property is a requirement for granulometric applications, and is implied by the increasing, extensivity and absorption properties [5]. Basically, it states that all features removed at a smaller scale will also be removed at a larger scale. This allows a sequence of operators of increasing size to ‘sieve’ the features in an image and classify them according to size (see Sect. 4).

Property 12. *Commutativity:*

$$\phi_{L^{(1)}}^{\triangleleft} \phi_{L^{(2)}}^{\triangleleft} f = \phi_{L^{(2)}}^{\triangleleft} \phi_{L^{(1)}}^{\triangleleft} f$$

This property follows from Property 10, and does not hold for the RIA dilation and RIA erosion.

Property 13. *Non-distributivity: Unlike the common dilation and erosion, the RIA dilation and erosion do not distribute with the extremum operators.*

Property 14. *Comparison with regular morphology:*

$$
\begin{aligned}
f \le \delta_L^{\triangleleft} f \le \delta_D f \\
f \ge \varepsilon_L^{\triangleleft} f \ge \varepsilon_D f \\
f \le \phi_L^{\triangleleft} f \le \phi_D f \\
f \ge \gamma_L^{\triangleleft} f \ge \gamma_D f
\end{aligned}
$$

4 Granulometry

Since the RIA closing and opening comply with the sieving property (Property 11), it is possible to use them as sieving functions in a granulometric application. A sieve is composed of a sequence of morphological filters with increasing size parameter [4]. The filters are applied either in series or in parallel (which produces the same result due to Property 10, absorption), each one removing a group of image features of certain size. This size is directly proportional to the filter parameter, and the measure that determines this size depends on the filter construction. Because of the sieving property, each filter removes all image features also removed by the smaller filters, and never adds new ones.

The difference between the result of subsequent filters is called a *granule image* [7], and contains only image features in a known size range. These granule images can be used to construct a size distribution. As said before, the measure used to determine the size of an image feature depends on the filter used. A closing with an isotropic structuring element (disk) measures the width of dark features. A RIA closing measures the length of dark features. Openings do the same with light features.

The set of granule images form a scale-space, which allows to measure the size of the feature that each pixel belongs to. The 'trace' of a pixel through the scales is some sort of local size distribution, which gives (for example through a mean or median) a scale parameter for that pixel. By going through this process with different filter types, we can assign different scale parameters to each pixel; for example the length and width of the pore that it belongs to. Knowing these values, it is easy to construct a distribution for the elongation.

5 Conclusions

We have defined some new morphological operators, based on the premise that, by dropping one or more dimensions, an isotopic structuring element in a subspace becomes anisotropic in the full image space, but also gains some degrees of rotational freedom. This freedom can be used to have the structuring element align itself to the features in the image, and thus become rotation invariant.

We have shown that the dilation with such a structuring element, giving it the orientation that causes the result to be maximal, is in fact an isotropic dilation. This comes from the fact that the isotropic structuring element is the same as the union of (an infinite amount of) lower-dimensional isotropic structuring elements with all possible different orientations. In contrast, if we give the structuring element the orientation that causes the result of the operation to be minimal, we get the dilation operator proposed here.

In the same manner, we have defined a new erosion, closing and opening operators. We have stated that all of these operators are rotation, translation, scaling and contrast invariant, as well as increasing and extensive. We have also mentioned that the closing and opening defined in this article are idempotent, commutative and absorbing. These properties are important if we want to use the new operators in the same way we use other morphological operators.

The morphological framework proposed in this article has been defined in two dimensions, but it has been shown that it is easy to extend to higher dimensional spaces. In two dimensions, the closing and opening as defined here can be used to measure the length of image features. In three dimensions, different versions of the same operator can measure both the first and second largest diameters. The smallest diameter is measured in all cases using an isotropic structuring element.

Finally, we have explored an example application for the new operators, that shows that they are useful in granulometric applications.

References

[1] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, London, 1982.

[2] C.L. Luengo Hendriks and L.J. van Vliet. A rotation-invariant morphology using anisotropic structuring elements. *in preparation*.

[3] R. van den Boomgaard. *Mathematical Morphology: Extensions Towards Computer Vision*. Ph.d. thesis, University of Amsterdam, Amsterdam, 1992.

[4] P. Soille. *Morphological Image Analysis*. Springer-Verlag, Berlin, 1999.

[5] G. Matheron. *Random Sets and Integral Geometry*. John Wiley and Sons, New York, 1975.

[6] P. Soille. Morphological operators with discrete line segments. In G. Borgefors, I. Nyström, and G. Sanniti di Baja, editors, *DGCI 2000, 9th Discrete Geometry for Computer Imagery*, volume 1953 of *LNCS*, pages 78–98, Uppsala, Sweden, 2000.

[7] J.A. Bangham, P. Chardaire, C.J. Pye, and P.D. Ling. Multiscale nonlinear decomposition: The sieve decomposition theorem. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 18(5):529–539, 1996.